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ON ASCHBACHER'S LOCAL $C(G;T)$ THEOREM

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For John Thompson

ABSTRACT

Aschbacher's local $C(G;T)$ theorem asserts that if G is a finite group with $F^*(G) = O_2(G)$, and $T \in \mathrm{Syl}_2(G)$, then $G = C(G;T)K(G)$, where $C(G;T) = \langle N_G(T_0) \mid 1 \neq T_0 \text{ char } T\rangle$ and $K(G)$ is the product of all near components of G of type $L_2(2^n)$ or A_{2^n+1}. Near components are also known as χ-blocks or Aschbacher blocks. In this paper we give a proof of Aschbacher's theorem in the case that G is a K-group, i.e., in the case that every simple section of G is isomorphic to one of the known simple groups. Our proof relies on a result of Meierfrankenfeld and Stroth [MS] on quadratic four-groups and on the Baumann-Glauberman-Niles theorem, for which Stellmacher [St2] has given an amalgam-theoretic proof. Apart from those results, our proof is essentially self-contained.

TABLE OF CONTENTS

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Introduction

It is especially fitting to honor John Thompson by writing on a topic of local group-theoretic analysis. Indeed, in the course of his verification of the Frobenius conjecture on fixed-point-free automorphisms, his proof with Feit of the solvability of groups of odd order, and his classification of N-groups (in particular, of the minimal simple groups), Thompson pioneered the development of the local techniques that were to dominate the classification of the finite simple groups over the next 20 years.

Prominent among the many ideas introduced by Thompson in these papers is the notion of the factorization or failure of factorization of a finite group G as the product of the normalizers (or centralizers) of suitable pairs of characteristic subgroups of a given Sylow subgroup of G. Aschbacher's local $C(G;T)$ theorem, a high point of local group theory during the final years, can be viewed as the ultimate extension of Thompson's ideas in one direction. Moreover, this deep result played a critical role in the classification of simple groups of characteristic 2 type.

By definition, for any subgroups X and R of the finite group G with R a 2-group,

$$C(X;R) = \langle N_X(R_0) \mid 1 \neq R_0 \text{ char } R\rangle.$$

Aschbacher's theorem [As2] asserts the following.

THEOREM (Aschbacher): *Let G be a group with $F^*(G) = O_2(G)$, and let $T \in \mathrm{Syl}_2(G)$. Then $G = K(G)C(G;T)$, where $K(G)$ is the product of all near components of G of type $L_2(2^n)$ or A_{2^n+1}.*

By definition, a **near component of G of type $L_2(2^n)$ or A_n** (also called a χ-block or Aschbacher block) is a subnormal subgroup L of G with the following properties:

(1) $F^*(L) = O_2(L)$ and $L = O^2(L)$;

(2) $L/O_2(L) \cong L_2(2^n)'$, $n \geq 1$, or A_n, $n \geq 5$ or $n = 3$;

(3) If $V = \Omega_1(Z(O_2(L)))$, then L centralizes $O_2(L)/V$; and

(4) If $U = [V, L]$ ($= [V, L/O_2(L)]$), then correspondingly $U/C_U(L)$ is the natural 2-dimensional $\mathbf{F}_{2^n}$-module or the unique nontrivial irreducible constituent of the permutation module for $L/O_2(L)$ (referred to as the standard A_n-module).

Remark: Near components L of type $L_2(2)$ and A_3 are isomorphic as $L_2(2)' \cong A_3 \cong Z_3$, and in either case $[V, L] \cong E_4$. On the other hand, even though $L_2(4) \cong A_5$, near components of these types are not isomorphic inasmuch as the corresponding modules are not quasi-equivalent. ∎

The subgroup $K(X)$ of any group X may be defined as in the theorem, and is clearly a characteristic subgroup of X. (If no near components of X of the appropriate types exist, we set $K(X) = 1$.)

For brevity, it is convenient to introduce the following terminology for any T-invariant subgroup G_1 of G containing $O_2(G)$: We say that G_1 has **Aschbacher form (relative to T)** provided $G_1 = K(G_1)C(G_1;T)$.

By a K-group we mean a group all of whose simple sections are isomorphic to known simple groups. However, Aschbacher does not assume that G is a K-group, instead allowing G to be an arbitrary finite group with $F^*(G) = O_2(G)$. But as a result he is forced to adopt a somewhat intricate strategy, based on a deep preliminary nongenerational result [As1], which is critical for dealing with a minimal configuration arising in the proof of his local $C(G;T)$ theorem. In particular, to identify the groups which appear in the conclusion of this prior result (namely, the groups $SL_n(2^m)$ and Σ_n), Aschbacher invokes two classification theorems: one by McLaughlin [Mc] on groups generated by transvections and the other by Timmesfeld [Ti] on groups with a weakly closed T.I. set.

Furthermore, an integral part of his argument in [As2] is a form of L-balance for near components, the proof of which is somewhat technical and, in particular, requires a third classification theorem, due to Stellmacher [St1], on groups generated by a conjugacy class of elements of order 3. In the original classification proof, this generalized L-balance was used along with the local $C(G;T)$ theorem in the analysis of simple groups of characteristic 2 type.

On the other hand, for applications to the classification of the simple groups, the local $C(G;T)$ theorem is needed only in the case that G is a K-group. Moreover, a revised strategy exists in which Aschbacher's general L-balance result for near components is not required. Under such a K-group assumption, it is in fact possible to give a considerably more direct proof of the local $C(G;T)$ theorem by abstracting relevant portions of Aschbacher's argument in [As2], completely avoiding [As1] as well as L-balance for near components (and thus reference to [St1]).

Indeed, except when certain critical sections of G are groups of Lie type of odd characteristic, and a few other groups, inductive arguments enable one to eliminate all but those configurations which lead to the desired conclusions. On the other hand, the case of Lie type of odd characteristic can be eliminated by appeal to a recent short elegant result of Meierfrankenfeld and Stroth [MS] concerning $\mathbf{F}_2$-modules admitting quadratic four-groups. Their paper represents a considerable simplification of Aschbacher's earlier treatment [As3] of such failure of Thompson factorization modules.

It should also be noted that Aschbacher handles minimal configurations in the local $C(G;T)$ theorem involving the groups $L_2(2^n)$ by reference to a fundamental theorem of Baumann-Glauberman-Niles ([B], [GN], [Ni]). One can now appeal to a simplified proof of their result due to Stellmacher [St2], based on the amalgam method.

In view of the critical role of the K-group version of the local $C(G;T)$ theorem in the classification of groups of characteristic 2 type (and, more generally, in the classification of groups of even type), it is important to have available an easily accessible proof of this result; and it is the purpose of this paper to provide such a treatment.

Thus we shall prove

THEOREM A: *If G is a K-group with $F^*(G) = O_2(G)$, then G has Aschbacher form (relative to a Sylow 2-subgroup).*

Our contributions here are limited primarily to the observation that the K-group assumption enables one to give a straightforward proof of the theorem, and to the organization and exposition of the resulting argument.

Indeed, essentially every minimal configuration we encounter has been treated by Aschbacher in [As2]. Our aim has been to make this paper reasonably self-contained and so we include complete arguments for each such minimal configuration, occasionally achieving some simplification. We also include a treatment of failure of factorization for the solvable case, which is a very minor variation of a theorem of Glauberman [Gl1].

We recall the fundamental configuration associated with the analysis of Thompson factorization [Th]. Let X be a group such that $F^*(X) = O_2(X)$. An elementary abelian 2-subgroup V of X is 2-reduced (in X) if and only if

$$V \triangleleft X \text{ and } O_2(X/C_X(V)) = 1.$$

Let $T \in \mathrm{Syl}_2(X)$ and set $Z = \Omega_1(Z(T))$. Then such subgroups V exist; for instance, $V = \langle Z^X \rangle$ is 2-reduced in X. For any 2-reduced subgroup V, $V \leq T$ as $V \triangleleft X$; and the study of Thompson factorization involves the following groups:

$$Z, \quad J = J(T), \quad E = \Omega_1(Z(J(T))), \quad S = C_T(E), \quad Q = O_2(X),$$
$$C = C_X(V), \text{ and } \bar{X} = X/C.$$

Here $J(T) = \langle \mathcal{A}(T) \rangle$, and $\mathcal{A}(T)$ is the set of all elementary abelian subgroups of T of maximal rank. The subgroup S is called the **Baumann subgroup** of T.

We refer to all these subgroups and factor groups as constituting a **2-reduced setup** (in X), and we use this notation for such a setup wherever possible.

In a 2-reduced setup, since $O_2(\bar{X}) = 1$, we have

$$Q \leq C \text{ and } V \leq \Omega_1(Z(Q)).$$

The 2-reduced subgroup V (or its setup) is called **singular** if and only if

$$\bar{J} \neq 1.$$

If $Z \leq V$ and V is not singular, then Thompson factorization holds, that is, $X = C_X(Z)N_X(J)$. Indeed, $J = J(T \cap C) \triangleleft N_X(T \cap C)$, $C = C_X(V) \leq C_X(Z)$, and $X = CN_X(T \cap C)$ by the Frattini argument. Consequently, in a counterexample G to the theorem, $V = \langle Z^G \rangle$ gives a singular 2-reduced setup.

Finally, we give a brief outline of this paper. §§1–7 can be viewed as preparatory to the proof of Theorem A. In particular, in §1 we state the two key results on which the proof depends from [MS] and [St2]. §2 includes a basic property of near components. In §§3 and 4 we study groups X having a singular 2-reduced setup for various K-groups $\bar{X}$, while in §5 we establish generational results for simple K-groups needed for our inductive argument. Then in §§6 and 7 we establish Baumann's critical $L_2(2^n)$ lemma and Aschbacher's Σ_{2n+1} analogue. These results are used in the proof of Theorem A to reduce direct products of $L_2(2^n)$'s and of A_{2n+1}'s to the case of a single factor.

With these results at our disposal we consider a minimal counterexample G to Theorem A in §§8, 9, and 10. By assumption, G is a K-group with $F^*(G) = O_2(G)$. We consider the 2-reduced subgroup $V = \langle Z^G \rangle$ and its setup, with notation as above. As noted there, if the corresponding setup is not singular, then $G = C_G(Z)N_G(J)$, so $G \leq C(G;T)$. Therefore, the setup is singular, so the results of §§3-7 apply to G. The minimality of G implies the following assertion:

(*) If $Q \le H$ and H is T-invariant, then H has Aschbacher form (relative to T) if and only if $HT < G$.

Using (*), we argue in §8 that if H is any normal subgroup of G containing Q with $HT < G$, then necessarily $Q \in \mathrm{Syl}_2(H)$. As an immediate corollary, it follows first that $Q \in \mathrm{Syl}_2(C)$ and second that either $F^*(\bar{G}) = E(\bar{G})$ is a product of isomorphic components transitively permuted by $\bar{T}$ or else $F^*(\bar{G}) = F(\bar{G}) \le O(\bar{G})$.

In §9, on the basis of the earlier analysis, we reduce the possibilities for the structure of $\bar{G}$ further; namely, we force $\bar{G}$ to have one of the following forms: $\bar{G} \cong L_2(2^n)$, $n \ge 1$, or Σ_{2n+1}, $n \ge 2$, or A_7. Moreover, in the second case, G has a near component of type A_{2n+1}. In the third case, G has what we shall call a **near component of small A_7-type**, namely, a subnormal subgroup L satisfying conditions (1) and (3) in the definition of near component above, but with $L/O_2(L) \cong A_7$ and $U = [V, L] \cong E_{16}$.

Finally, in §10 we rule out the third case and show that G has Aschbacher form in the two residual cases, a contradiction, thus completing the proof of Theorem A.

We extend our gratitude to both Professor Thompson and the referee, who pointed out several errors and obscurities in our manuscript and suggested useful changes.

1. Assumed results

Throughout the paper X will denote a K-group with $F^*(X) = O_2(X)$, so that X has a 2-reduced setup (as stipulated in the Introduction, with the accompanying notation).

In this section we state the two key results on which the proof depends. The first is the Baumann–Glauberman–Niles theorem, for which we use [St2] as reference.

THEOREM 1.1: *Assume $Q \in \mathrm{Syl}_2(C)$ and set $\tilde{X} = X/Q$. If $\tilde{X}/\Phi(\tilde{X}) \cong L_2(2^n)$, $n \ge 1$, and no nontrivial characteristic subgroup of T is normal in X, then $O^2(X)$ is a near component of X of type $L_2(2^n)$.*

Next we state the Aschbacher–Meierfrankenfeld–Stroth theorem [MS].

THEOREM 1.2: *Assume that $\bar{L} = F^*(\bar{X})$ is a quasisimple group of Lie type of odd characteristic, and that $\bar{L} \notin Chev(2)$. If $\bar{X}$ contains a four-group $\bar{U}$ such that $[V, \bar{U}, \bar{U}] = 1$, then $\bar{L} \cong 3U_4(3)$ or $[3 \times 3]U_4(3)$.*

Remark: Any elementary abelian 2-subgroup $\bar{U}$ of $\bar{X}$ such that $[V, \bar{U}, \bar{U}] = 1$ is said to act quadratically on V. In the case that X has a singular 2-reduced setup, Thompson's replacement theorem (cf. [Go; 8.25]) implies the existence of $A \in \mathcal{A}(T)$ such that $\bar{A} \neq 1$ and $\bar{A}$ acts quadratically on V. See Lemma 3.1. ∎

Beyond these results, the proof of Theorem A depends on standard results of local analysis, including the Baer–Suzuki theorem and Glauberman's Z^*-theorem [Gl2]. In addition, we require generational results for known simple groups other than those of Lie type of odd characteristic, and also more detailed properties of certain specific (families of) simple groups — including the basic modules for $L_2(2^n)$, $(S)L_3(2^n)$, and $Sp_4(2^n)$ over algebraically closed fields of characteristic 2, in conjunction with Steinberg's tensor product theorem.

2. Products of near components

In this section we establish the following important property of near components [As2; 3.4]. As stipulated in §1, X is a K-group with $F^*(X) = O_2(X)$; however, only the latter assumption is needed here.

PROPOSITION 2.1: *Distinct near components of X centralize each other.*

Proof: Let K, L be distinct near components of X. Also set $Q = O_2(X)$ and $\bar{X} = X/Q$. By the subnormality of K and L, $O_2(K)$ and $O_2(L)$ are contained in Q, and as $K = O^2(K)$, we have $K = O^2(KQ)$ and similarly $L = O^2(LQ)$. In particular, K and L have unique noncentral chief factors in their action on Q, and $\bar{K} \neq \bar{L}$. Furthermore, if K is nonsolvable, then as $\bar{K}$ is subnormal in $\bar{X}$ and quasisimple, $\bar{K}$ is a component of $\bar{X}$. On the other hand, if K is solvable, we conclude by induction on the length of a normal series from K to X that $K \leq O_{23}(X)$, whence $\bar{K} \leq O_3(\bar{X})$. Similar statements hold for $\bar{L}$.

Suppose first that, say, K is nonsolvable. Then $\bar{K}$ centralizes $O_3(\bar{X})$, so by the preceding paragraph $\bar{K}$ centralizes $\bar{L}$ whether $\bar{L}$ is nonsolvable or solvable. Thus K and L normalize each other, and $[L, K] \leq Q$. Set $W = [K, Q]$. Then K/W is quasisimple so $[L, K] \leq W$. If L is solvable, then $L \cong A_4$ and obviously K centralizes L, so we can assume that L is nonsolvable. Since L acts on $W/C_W(K)$

and commutes with the irreducible action of K on the module, $L = L'$ centralizes $W/C_W(K)$. But as noted above, $[L,K] \le W$, so $W_0 = [L,K,L] \le C_W(K)$. Thus W_0 is normalized by K as well as L. Setting $Y = KLW_0$ and $\tilde{Y} = Y/W_0 = \tilde{K}\tilde{L}$, it follows that $[\tilde{K}, \tilde{L}, \tilde{L}] = 1$. Since $\tilde{L}$ is perfect, the three subgroups lemma yields that $[\tilde{K}, \tilde{L}] = 1$, so $[K,L] \le W_0 \le C_W(K)$. Hence $[K,L,K] = 1$. Another application of the three subgroups lemma shows that $[K,L] = 1$ as K is perfect. Thus the proposition holds if K is nonsolvable; by symmetry it also holds if L is nonsolvable.

Suppose finally that K and L are solvable. Let $\langle x \rangle \in \mathrm{Syl}_3(K)$ and $\langle y \rangle \in \mathrm{Syl}_3(L)$ and set $\bar{R} = \langle \bar{x}, \bar{y} \rangle = \langle \bar{K}, \bar{L} \rangle$. By what we have shown above, $\bar{R} \le O_3(\bar{X})$. Now $Z(Q) \ge W = O_2(K) \cong E_4$ and K centralizes Q/W. Likewise $Z(Q) \ge V = O_2(L) \cong E_4$ and L centralizes Q/V. Thus K and L normalize VW and centralize Q/VW. Since $F^*(X) = Q$, the 3-group $\bar{R}$ therefore acts faithfully on VW. But $|VW| \le 16$ and so $\bar{R} \le E_9$ (as $GL_4(2)$ has E_9 Sylow 3-subgroups). Since $\bar{K} \ne \bar{L}$, $\bar{R} \cong E_9$.

If $V = W$, then some element of $\bar{R}^{\#}$ centralizes Q, contrary to $F^*(X) = Q$. It follows that $V \cap W = 1$, so $VW = V \times W$. But $\bar{K}\bar{L} = \bar{K} \times \bar{L}$, so again K normalizes L and L normalizes K. Since $K \cong L \cong A_4$ and $V \cap W = 1$, $K \cap L = 1$. Hence $[K,L] \le K \cap L = 1$, so the proposition holds in this case as well.

3. Failure of Thompson factorization

In this section we establish the results we need concerning failure of Thompson factorization for the group X. We assume throughout that X has a singular 2-reduced setup (as agreed, X is a K-group with $F^*(X) = O_2(X)$ with the accompanying notation as in the Introduction).

In this situation, V is an $\mathbf{F}_2\bar{X}$-module, $\bar{J} \ne 1$ by assumption, and the following basic facts are fundamental for studying the action of $\bar{J}$ on V.

LEMMA 3.1: *Let $A \in \mathcal{A}(T)$ with $\bar{A} \ne 1$, and set $A_0 = A \cap C$ and $V_0 = C_V(A)$. Then the following conditions hold:*

- (i) *$|V : V_0| \le |\bar{A}|$. In particular, if $\bar{A} = \langle \bar{a} \rangle \cong Z_2$, then equality holds and $\bar{a}$ induces a transvection on V;*
- (ii) *If equality holds in (i), then $A_0V \in \mathcal{A}(T)$;*
- (iii) *$A_0 \cap V \le V_0$;*
- (iv) *For any $\bar{a} \in \bar{A}^{\#}$, $m([V,\bar{a}]) = m(V/C_V(\bar{a}))$ and $V_0 \le C_V(\bar{a})$; and*

(v) *If there is no* $A_1 \in \mathcal{A}(T)$ *such that* $1 \neq \bar{A}_1 < \bar{A}$, *then*

(1) A *acts quadratically on* V *(that is,* $[V, A] \leq V_0$*); and*

(2) *If* $|\bar{A}| > 2$, *then* $V_0 = C_V(B)$ *for every hyperplane* B *of* A.

Proof: Since A is abelian, A centralizes $A \cap V = A_0 \cap V$, so $A_0 \cap V \leq V_0$, proving (iii). But VA_0 is elementary abelian, so $m(VA_0) \leq m(A) = m(A_0) + m(\bar{A})$. However, $m(VA_0) = m(V) + m(A_0) - m(V \cap A_0) \geq m(V) + m(A_0) - m(V_0)$, so $m(\bar{A}) \geq m(V) - m(V_0) = m(V/V_0)$.

Also, if $m(\bar{A}) = 1$, then as $\bar{A}$ acts faithfully on V, $m(V/V_0) = 1$, so $\bar{A} = \langle \bar{a} \rangle$ centralizes a hyperplane of V and hence induces a transvection on V. Thus (i) holds. Furthermore, if equality holds in (i), the preceding calculation shows that $m(VA_0) = m(A)$, whence $VA_0 \in \mathcal{A}(T)$, proving (ii).

As for (iv), obviously $V_0 = C_V(A) \leq C_V(\bar{a})$ for any $\bar{a} \in \bar{A}^{\#}$. Furthermore, as $\bar{a}$ is an involution, it is immediate that $m(V/C_V(\bar{a})) = m([V, \bar{a}])$, so (iv) holds.

Finally, as V is abelian, the Thompson replacement theorem implies (v1). Indeed, if (v1) fails, then $[V, A, A] \neq 1$, so $[B, A, A] \neq 1$, where $B = VA_0$. By that theorem there is $A^* \in \mathcal{A}(BA)$ such that $|A^* \cap B| > |A \cap B|$ and $[A^*, A, A] = 1$. Since $[B, A, A] \neq 1$, $A^* \neq B$, so $A^* \not\leq B$. These conditions imply $1 \neq |\overline{A^*}| < |\bar{A}|$, contrary to assumption. Thus (v1) holds. Suppose that (v2) fails for some $\bar{B}$ of index 2 in $\bar{A}$. Then $\bar{B} \neq 1$ as $|\bar{A}| > 2$ by assumption and if $V_1 = C_V(\bar{B})$, then $V_0 < V_1$. Let B be the preimage of $\bar{B}$ in A, so that BV_1 is elementary. By our choice of A, $BV_1 \in \mathcal{A}(T)$. However, as $\bar{B}\bar{V}_1 = \bar{B} < \bar{A}$, this contradicts our hypothesis on A. Hence (v2) also holds, and the lemma is proved. [The replacement theorem [Go; 8.2.5] is stated for a definition of $\mathcal{A}(T)$ and $J(T)$ slightly different from our definitions here, but the same result is true for our definitions, assuming that B is elementary abelian, as is the case here. The proof is the same.]

We also need the following general result here and later in the paper about involutions of $\bar{X}$ acting on subgroups of $\bar{X}$ of odd order. To state it, for any cyclic group $Y = \langle y \rangle$ of odd order, define $d(Y) = d(y)$ to be the smallest degree of a faithful module over $\mathbf{F}_2$ for the dihedral group of order $2|Y|$.

LEMMA 3.2: *Let* $\bar{t}$ *be an involution of* $\bar{X}$ *and* $\bar{Y}$ *a nontrivial* $\bar{t}$*-invariant subgroup of* $\bar{X}$ *of odd order such that* $\bar{Y} = [\bar{Y}, \bar{t}]$. *Then the following conditions hold:*

(i) *If* $\bar{Y}$ *is cyclic, then* $m([V, \bar{t}]) \geq \frac{1}{2} d(\bar{Y})$;

(ii) *If* $\bar{t}$ *induces a transvection on* V, *then*

(1) $\bar{Y} \cong Z_3$;
(2) *If* $\bar{Y} \leq O(\bar{X})$, *then* $\bar{t} \in O_{2'2}(\bar{X})$; *and*
(3) $\bar{t}$ *leaves invariant every component of* $\bar{X}$;

(iii) *If* $\bar{Y} \cong 3^{1+2}$ *and we set* $W = [V, Z(\bar{Y})]$, *then* $m(W) = 6r$ *for some* $r \geq 1$ *and* $m([W, \bar{t}]) \geq 2r$;

(iv) *If* $\bar{Y}$ *is a 3-group with* $\bar{Y}/\Phi(\bar{Y}) \cong E_{81}$ *and* $m([V, \bar{t}]) \leq 2$, *then* 3^{1+4} *is not a homomorphic image of* $\bar{Y}$; *and*

(v) *If* $\bar{Y} = Z(\bar{Y}_1)$, *where* $\bar{Y}_1 \cong 3^{1+2n}$, *then* $m([V, \bar{t}]) \geq 3^n$.

Proof: Let $k = m([V, \bar{t}])$ and $V_0 = C_V(\bar{t})$. Since $\bar{t}$ is an involution, it follows that $m(V/V_0) = k$. Suppose (i) fails and set $\bar{Y} = \langle \bar{y} \rangle$. Then $m(V/V_0) < \frac{1}{2}d(\bar{Y})$ and hence $m(V/U) < d(\bar{Y})$, where $U = V_0 \cap V_0^{\bar{y}}$. But $\bar{t}$ inverts $\bar{Y}$, so $\bar{Y}\langle \bar{t} \rangle = \langle \bar{t}, \bar{t}^{\bar{y}} \rangle$. However, the latter group centralizes U, so $\bar{Y}$ centralizes U and therefore $\bar{Y}\langle \bar{t} \rangle$ acts faithfully on V/U, which has dimension $n < d(\bar{Y})$. This contradicts the definition of $d(\bar{Y})$, so (i) holds.

Suppose now that $\bar{t}$ acts as a transvection on V, so that $k = 1$ and $m(V/V_0) = 1$. Hence by (i), $\bar{t}$ inverts no element of $\bar{Y}$ of order > 3. Thus if (ii1) fails and $\bar{Y}$ is chosen minimal to violate its conclusion, then $\bar{Y}$ is a 3-group and $\bar{Y}/\Phi(\bar{Y}) \cong E_9$ is inverted by $\bar{t}$. Therefore $\bar{Y} = \langle \bar{y}_1, \bar{y}_2 \rangle$ with $\bar{y}_i$ inverted by $\bar{t}$, $i = 1, 2$, and $\bar{Y}\langle \bar{t} \rangle = \langle \bar{t}, \bar{t}^{\bar{y}_1}, \bar{t}^{\bar{y}_2} \rangle$. This time, setting $U = V_0 \cap V_0^{\bar{y}_1} \cap V_0^{\bar{y}_2}$, it follows that $m(V/U) \leq 3$, and again $\bar{Y}$ centralizes U and acts faithfully on V/U. However, $\mathrm{Aut}(V/U) \leq GL_3(2)$, which has Sylow 3-subgroups of order 3, contrary to $|\bar{Y}| \geq 9$. Thus (ii1) also holds.

Now assume that (ii2) fails and set $\tilde{X} = \bar{X}/O_{2'2}(\bar{X})$, so that $\tilde{t}$ is an involution and $O_2(\tilde{X}) = 1$. By the Baer-Suzuki theorem [Go; 3.8.2], $\tilde{t}$ inverts an element $\tilde{y}_1 \in \tilde{X}$ of odd prime order. Then $\bar{t}$ inverts an element $\bar{y}_1$ of odd order in $\bar{X}$ mapping on $\tilde{y}_1$. Hence if we set $\bar{Y}_1 = \langle \bar{Y}, \bar{y}_1 \rangle$, then $\bar{Y}_1 = [\bar{Y}_1, \bar{t}]$. But $\bar{Y}_1 > \bar{Y}$ as $\bar{Y} \leq O(\bar{X})$ by assumption. Thus $|\bar{Y}_1| > 3$, contrary to (ii1). Therefore (ii2) also holds.

Suppose that $\bar{t}$ does not leave invariant some component $\bar{L}$ of $\bar{X}$. Since $\bar{L}$ is not a $\{2,3\}$-group, $\bar{t}$ inverts an element $\bar{y}$ of $\bar{L}\bar{L}^{\bar{t}}$ of prime order $p \geq 5$. But then (ii1) is contradicted with $\bar{Y} = \langle \bar{y} \rangle$, so (ii3) also holds.

Next, assume $\bar{Y}$ and W are as in (iii), so that $Z(\bar{Y})$ has no nontrivial fixed points on W. In particular, every chief factor of $W\bar{Y}$ within W is a faithful irreducible $\bar{Y}$-module. Since $\bar{Y} \cong 3^{1+2}$, each such chief factor has rank 6, so $m(W) = 6r$, where V is the number of such factors.

Let $W_1 \le W$ be irreducible as $\bar{Y}$-module. Now W is $\bar{t}$-invariant, so $W_1^{\bar{t}} \le W$. Then either $W_1 \cap W_1^{\bar{t}} = 1$ or $W_1^{\bar{t}} = W_1$. Proceeding by induction on r, it suffices to prove (iii) for $W_1 W_1^{\bar{t}}$. If $W_1 \cap W_1^{\bar{t}} = 1$, then $\bar{t}$ interchanges W_1 and $W_1^{\bar{t}}$, whence $m([W_1 W_1^{\bar{t}}, \bar{t}]) = 6$, in which case the desired conclusion clearly holds. On the other hand, if $\bar{t}$ leaves W_1 invariant, we need only show that $m([W_1, \bar{t}]) \ge 2$. However, if this fails, then $\bar{t}$ induces a transvection on W_1 and hence (ii) fails (with W_1 in place of V). Thus (iii) also holds.

Next, let $\bar{Y}$ be as in (iv) with $m([V, \bar{t}]) \le 2$, whence $m(V/V_0) \le 2$. Since $[\bar{Y}, \bar{t}] = \bar{Y}$, $\bar{t}$ inverts $\bar{Y}/\Phi(\bar{Y}) \cong E_{81}$, so $\bar{Y} = \langle \bar{y}_i \mid 1 \le i \le 4 \rangle$, where each $\bar{y}_i$ is inverted by $\bar{t}$. This time setting $U = V_0 \cap V_0^{\bar{y}_1} \cap V_0^{\bar{y}_2} \cap V_0^{\bar{y}_3} \cap V_0^{\bar{y}_4}$, we conclude as above that $\bar{Y}$ acts faithfully on V/U and that $m(V/U) \le 10$, so $\bar{Y}$ embeds in $GL_{10}(2)$. However, one checks directly that a Sylow 3-subgroup of $GL_{10}(2)$ has an abelian subgroup of index 3, so does not involve 3^{1+4}, and (iv) also holds.

Finally, in (v), a faithful irreducible $\mathbf{F}_2\bar{Y}_1$-module V_1 has dimension $2 \cdot 3^n$, and is acted on without fixed points by $\bar{Y} = Z(\bar{Y}_1)$. Since $\bar{t}$ inverts $\bar{Y}$, $\dim[\bar{t}, V_1] = 3^n$, which implies (v).

Next we state a reduction lemma which will be used frequently.

LEMMA 3.3: *Assume that $A \in \mathcal{A}(T)$, $A \le B \le T$, and $\bar{W}$ is a $\bar{B}$-invariant subgroup of $\bar{X}$ such that $O_2(\bar{W}) \le Z(\bar{W})$. Set $\bar{X}_1 = \bar{B}\bar{W}$, $\bar{R} = O_2(\bar{X}_1)$, and $V_1 = C_V(\bar{R})$, and let X_1 be the preimage of $\bar{X}_1$ in X. Then the following conditions hold:*

(i) *V_1 is 2-reduced in X_1;*
(ii) *If we put $C_1 = C_{X_1}(V_1)$ and $\tilde{X}_1 = X_1/C_1$, then $\bar{C}_1 = \bar{R}$, $F^*(\tilde{X}_1) = F^*(\tilde{W})$ and $\tilde{W} \cong \bar{W}/O_2(\bar{W})$; and*
(iii) *If $[\bar{A}, \bar{W}] \not\le O_2(\bar{W})$, then V_1 is singular in X_1.*

Proof: Our hypotheses imply that $[F^*(\bar{W}), O_2(\bar{X}_1)] \le O_2(\bar{W}) \le Z(\bar{W})$. Therefore $[F^*(\bar{W}), O_2(\bar{X}_1), F^*(\bar{W})] = 1$, and so every element of $F^*(\bar{W})$ of odd order centralizes $O_2(\bar{X}_1)$. By the $A \times B$ lemma, it follows that such an element of odd order centralizes V_1 if and only if it is the identity. Hence $\bar{C}_1 \cap F^*(\bar{W}) = C_{F^*(\bar{W})}(V_1)$ is a 2-group. But clearly $O_2(\bar{W}) \le O_2(\bar{X}_1) \le \bar{C}_1$, so $\bar{C}_1 \cap F^*(\bar{W}) = O_2(\bar{W})$. Furthermore, $F^*(\bar{W})/O_2(\bar{W}) = F^*(\bar{W}/O_2(\bar{W}))$ since $O_2(\bar{W}) \le Z(\bar{W})$. Therefore $(\bar{C}_1 \cap \bar{W})/O_2(\bar{W})$ is a normal subgroup of $\bar{W}/O_2(\bar{W})$ disjoint from $F^*(\bar{W}/O_2(\bar{W}))$, so $\bar{C}_1 \cap \bar{W} = O_2(\bar{W})$. It follows that $\tilde{W} \cong \bar{W}/O_2(\bar{W})$ and that $\bar{C}_1$ is a 2-group, so $\bar{C}_1 = O_2(\bar{X}_1)$.

Therefore $O_2(\tilde{X}_1) = 1$, proving (i). Also $F^*(\tilde{X}_1) = O^2(F^*(\tilde{X}_1)) = F^*(\tilde{W})$ as $\bar{W} \geq O^2(\bar{X}_1)$. Now (ii) follows. Finally, if $[\bar{A}, \bar{W}] \not\leq O_2(\bar{W})$, then by (ii), $[\tilde{A}, \tilde{W}] \neq 1$, so (iii) holds and the lemma is proved.

Our first major result is due to Glauberman [Gl1]. (His statement is limited to the case $V = \langle Z^X \rangle$ and X is solvable, but the result holds for an arbitrary singular 2-reduced setup in which $\bar{X}$ is solvable.)

PROPOSITION 3.4: *If $\bar{X}$ is solvable, then there exists an integer $r \geq 1$ such that*

(i) $\langle \bar{J}^{\bar{X}} \rangle = \bar{L}_1 \times \cdots \times \bar{L}_r$ *with each* $\bar{L}_i \cong \Sigma_3$;

(ii) *For each i, $\bar{L}_i \cap \bar{T} = \langle \bar{a}_i \rangle$, where $\bar{a}_i$ acts as a transvection on V;*

(iii) $V = V_0 \times V_1 \times \cdots \times V_r$, *where* $V_0 = C_V(\langle \bar{J}^{\bar{X}} \rangle)$ *and* $V_i = [V, \bar{L}_i] \cong E_4$ *for* $1 \leq i \leq r$, *with $\bar{L}_i$ faithful on V_i and $[\bar{L}_i, V_j] = 1$ for all $i \neq j$; and*

(iv) *S normalizes each $\bar{L}_i$, $1 \leq i \leq r$.*

Proof: Observe that (iv) follows from the other parts. Namely, by (i) and the Krull-Schmidt theorem, $\bar{S}$ permutes the $\bar{L}_i$, and permutes the V_i in the same way. But by (i), $E \cap V_i \neq 1$ for all i, and so since $V_i \cap V_j = 1$ for $i \neq j$, $S = C_T(E)$ normalizes each V_i and hence each $\bar{L}_i$.

For the other parts, we proceed by induction on $|\bar{X}|$. Set $\bar{D} = O(\bar{X})$ and let $\bar{R} = \bar{T} \cap O_{2'2}(\bar{X})$. If $\bar{X} > O_{2'2}(\bar{X})$, then by induction $\langle \bar{J}^{\bar{T}\bar{D}} \rangle = \bar{L}_1 \times \cdots \times \bar{L}_r$ as in the proposition. In particular, $\langle \bar{J}^{\bar{T}\bar{D}} \rangle$ is generated by transvections on V, so lies in $O_{2'2}(\bar{X})$ by Lemma 3.2. Thus $\bar{J} \leq \bar{T} \cap O_{2'2}(\bar{X}) = \bar{R}$. Letting R be the preimage of $\bar{R}$ in T, we have by the Frattini argument that $\bar{X} = \overline{DN_X(R)}$. But $N_X(R)$ normalizes $J(R) = J$. Thus $\langle \bar{J}^{\bar{X}} \rangle = \langle \bar{J}^{\bar{D}} \rangle = \langle \bar{J}^{\bar{T}\bar{D}} \rangle$, and the result follows by induction. We may therefore assume that $\bar{X} = \bar{D}\bar{T}$. If $\bar{J} < \bar{T}$, induction in $\bar{D}\bar{J}$ yields the desired assertion. Thus we may also assume that $\bar{J} = \bar{T}$.

Let $\mathcal{A}_0(T) = \{A \in \mathcal{A}(T) \mid \bar{A} < \bar{T}\}$. We next treat the case

$$\bar{T} = \langle \bar{A} \mid A \in \mathcal{A}_0(T) \rangle. \tag{3.1}$$

Let $A \in \mathcal{A}_0(T)$ and let T_0 be a maximal subgroup of T such that $\bar{A} \leq \bar{T}_0$, set $\bar{X}_0 = \bar{D}\bar{T}_0$, and let X_0 be the preimage of $\bar{X}_0$ in X. Set $J_0 = J(T_0)$, so that $\bar{A} \leq \bar{J}_0$ and hence $\bar{J}_0 \neq 1$. Also set $\bar{W} = \langle \bar{J}_0^{\bar{D}} \rangle$, so that since $J_0 \triangleleft T$, $\bar{W} = \langle \bar{J}_0^{\bar{X}} \rangle$. By induction $\bar{W} = \bar{L}_1 \times \cdots \times \bar{L}_r$ as in the proposition. If $\bar{J} = \bar{J}_0$, we are done, so we may assume that $\bar{J}_0 < \bar{J}$, whence $\bar{J} \not\leq \bar{T}_0$.

We argue that $\bar{J}$ normalizes $\bar{L}_i$ for each $i = 1, 2, \ldots, r$. If false, then by (3.1), for some $B \in \mathcal{A}_0(T)$, $\bar{B}$ does not normalize $\bar{L}_1$, say. Since $\bar{B} < \bar{T}$, by induction in $\bar{D}\bar{B}$, $\bar{B}$ is generated by elements $\bar{b}$ acting on V as transvections. For any such $\bar{b}$, $C_V(\bar{b}) \cap [V, \bar{L}_i] \neq 1$ as $|[V : \bar{L}_i]| = 4$.

On the other hand, the groups $[V, \bar{L}_i]$ generate their direct product. Consequently, for any $\bar{x} \in \bar{W}$ of order 3, $[V, \bar{x}]$ is the product of certain $[V, \bar{L}_i]$, and $\bar{x} \in \bar{L}_i$ if and only if $[V, \bar{x}] = [V, \bar{L}_i]$, so the $[V, \bar{L}_i]$ are the minimal elements of the set $\{[V, \bar{x}] \,|\, \bar{x}^3 = 1 \neq \bar{x} \in \bar{W}\}$. As $\bar{W} \lhd \bar{X}$, each such $\bar{b}$ permutes this set and so permutes the $[V, \bar{L}_i]$. Hence by the preceding paragraph, each $\bar{b}$ normalizes all $[V, \bar{L}_i]$, whence $\bar{B}$ does as well. But then $\bar{B}$ normalizes $\{\bar{x} \in \bar{W} \,|\, \bar{x}^3 = 1, [V, \bar{x}] = [V, \bar{L}_1]\}$. However, because of the faithful action of $\bar{W}$ on V, this set is precisely $O_3(\bar{L}_1)^{\#}$. Therefore $\bar{B}$ normalizes $\bar{L}_1$, contradiction. This proves the assertion.

Further, if $\bar{a}_i \in \bar{L}_i$ is an involution, then since $\bar{a}_i$ is a transvection on V, Lemma 3.2 implies that $|[\bar{a}_i, \bar{D}]| = 3$, so $O_3(\bar{L}_i) = [\bar{a}_i, \bar{D}]$. Therefore $\bar{L}_i = \langle \bar{a}_i, [\bar{a}_i, \bar{D}] \rangle$ is normalized by $C_{\bar{D}}(\bar{a}_i)$ and hence by $\bar{D} = [\bar{a}_i, \bar{D}] C_{\bar{D}}(\bar{a}_i)$. We have proved that $\bar{L}_i \lhd \bar{D}\bar{J} = \bar{X}$ (assuming (3.1)).

Since $\bar{L}_i \cong \Sigma_3$, if follows that $\bar{X} = \bar{W} \times \bar{W}_1$, where $\bar{W}_1 = C_{\bar{X}}(\bar{W})$. In particular, as $\bar{A} \leq \bar{W}$, $\bar{A} \leq Z(\bar{T})$. Moreover, by induction, $\langle \bar{A}^{\bar{D}} \rangle = \langle \bar{A}^{\bar{X}} \rangle$ is the direct product of some of the $\bar{L}_i$, say $\bar{L}_1 \times \cdots \times \bar{L}_t$.

Furthermore, if B is any other element of $\mathcal{A}_0(T)$, we have similarly that $\langle \bar{B}^{\bar{D}} \rangle = \bar{M}_1 \times \cdots \times \bar{M}_s$, where all $\bar{M}_i \cong \Sigma_3$ and $\bar{M}_i \cap \bar{T}$ acts as a transvection on V. Since $[\bar{A}, \bar{B}] = 1$, it follows that each $\bar{M}_i$ either lies in $\bar{W}_1$ or is an $\bar{L}_j$, $1 \leq j \leq t$. Hence $\langle \bar{A}^{\bar{D}} \rangle \langle \bar{B}^{\bar{D}} \rangle$ is a direct product satisfying all the conclusions of the theorem and is a direct factor of $\bar{X}$.

By (3.1), $\bar{T} = \bar{J}$ is generated by the set of all $A \in \mathcal{A}_0(T)$. Hence repeating the above procedure for every pair of elements of $\mathcal{A}_0(T)$ yields the conclusion of the proposition. Thus it remains to treat the case in which (3.1) fails. Since $\bar{J} = \langle \bar{A} \,|\, A \in \mathcal{A}(T) \rangle$, it follows that $\bar{J} = \bar{T} = \bar{A}$ for some $A \in \mathcal{A}(T)$, whence $\bar{X} = \bar{D}\bar{A}$.

By the Thompson dihedral lemma [Th; p.409], there exists a subgroup $\bar{D}_0$ of $\bar{D}$ and a decomposition $\bar{A}\bar{D}_0 = \bar{D}_1 \times \cdots \times \bar{D}_n$, where $\bar{D}_i$ is dihedral of order $2p_i$(p_i an odd prime) and $n = m(\bar{A})$. Set $\bar{T} \cap \bar{D}_i = \langle \bar{a}_i \rangle$ and $\bar{A}^i = \langle \bar{a}_j \,|\, j \neq i \rangle$. Fix i and set $V_1 = C_V(\bar{A}^i)$; by the $A \times B$ lemma, $\bar{D}_i$ acts faithfully on V_1. Hence $C_V(\bar{A}^i) > C_V(\bar{A})$. Therefore if A^i is the preimage in A of $\bar{A}^i$, $m(A^i C_V(\bar{A}^i)) > m(A^i C_V(\bar{A}))$ (since $A^i \cap V \leq A \cap V \leq C_V(\bar{A})$). But $m(A^i) = m(A) - 1$ and so $B_i = A^i C_V(\bar{A}^i) \in \mathcal{A}(T)$. If $n > 1$, we have $\bar{T} = \bar{A}^1 \bar{A}^2 = \bar{B}_1 \bar{B}_2$ and $B_1, B_2 \in$

$\mathcal{A}_0(T)$, so (3.1) holds, contrary to assumption. We conclude that $n = 1$.

Thus $\bar{A} = \langle \bar{a} \rangle$. But in this case Lemma 3.2 implies that $\bar{a}$ acts as a transvection on V and yields the conclusion of the proposition. The proof is complete.

Our next result is the elimination of certain possibilities for the components of $\bar{X}$.

PROPOSITION 3.5: *Let $\bar{L}$ be a component of $\bar{X}$ not centralized by $\bar{J}$. Then the following hold:*

(i) *If $\bar{L} \in Chev(p)$ for some odd p, then also $\bar{L} \in Chev(2)$ (so $\bar{L}$ is an "ambiguous" group); and*

(ii) *$\bar{L} \not\cong Sz(2^n)$, $(S)U_3(2^n)$, $3A_7$, J_1, or Ly.*

We argue by contradiction and choose a counterexample with $\bar{X}$ of minimal order. Choose $A \in \mathcal{A}(T)$ such that $[\bar{A}, \bar{L}] \neq 1$ and, subject to this, so that $\bar{A}$ has minimal order.

We first prove

LEMMA 3.6: *The following conditions hold:*

(i) *$\bar{L}$ is $\bar{A}$-invariant and $\bar{X} = \bar{L}\bar{A}$;*

(ii) *$[\bar{A}, Z(\bar{L})] = 1$; and*

(iii) *A acts quadratically on V.*

Proof: First, if $\bar{W}$ is an $\bar{A}$-invariant subgroup of $\bar{X}$ such that $\bar{W} = E(\bar{W})$, $O_2(\bar{W}) = 1$, and $[\bar{W}, \bar{A}] \neq 1$, we can apply Lemma 3.3 with A in the role of B there. With $\tilde{W}$ as defined there, we have $\tilde{W} \cong \bar{W}$, and so either $\bar{X} = \bar{W}\bar{A}$ or else every component of $\tilde{W}$ satisfies the conclusions of the proposition, by our minimal choice. Hence so does every component of $\bar{W}$ not centralized by $\bar{A}$.

Taking $\bar{W} = \langle \bar{L}^{\bar{A}} \rangle$, we conclude that $\bar{X} = \langle \bar{L}^{\bar{A}} \rangle \bar{A}$, since $\bar{L}$ does not satisfy the conclusions of the proposition. Set $\bar{K} = \langle \bar{L}^{\bar{A}} \rangle$, so that $\bar{K} = F^*(\bar{X})$.

To prove (i), it suffices then to prove that $\bar{A}$ normalizes $\bar{L}$ — that is, $\bar{K} = \bar{L}$. Suppose false and choose $\bar{a} \in \bar{A} - N_{\bar{A}}(\bar{L})$, and apply the first paragraph to $\bar{W} = E(C_{\bar{K}}(\bar{a}))$. Since $F^*(\bar{X}) = \bar{K}$, $C_{\bar{X}}(\bar{W}) = \langle \bar{a} \rangle$. Hence if $[\bar{W}, \bar{A}] = 1$, we would have $\bar{A} = \langle \bar{a} \rangle$, and Lemma 3.2(ii3) would be violated. Thus $[\bar{W}, \bar{A}] \neq 1$, so the first paragraph yields that $\bar{L}$, a central extension of a component of $\bar{W}$, satisfies the conclusion of the proposition, or else $\bar{L} \cong 3A_7$ and the components of $\bar{W}$ are isomorphic to A_7.

However, in the latter case $\bar{a}$ inverts $Z(\bar{L})$, because $\bar{x} \mapsto \bar{x}\bar{x}^{\bar{a}}$ defines a surjection from $\bar{L}$ to a component of $\bar{W}$. The same holds for any $\bar{a} \in \bar{A} - N_{\bar{A}}(\bar{L})$, which implies that $|\bar{A}/N_{\bar{A}}(\bar{L})| = 2$, whence $\bar{K} = \bar{L}\bar{L}^{\bar{a}}$, so $m(\bar{A}) = 1 + m(N_{\bar{A}}(\bar{L})) \leq 1 + m_2(\mathrm{Aut}(\bar{L})) = 4$. Hence $m([V,\bar{a}]) \leq 4$ by Lemma 3.1. On the other hand, choosing $\bar{P} \in \mathrm{Syl}_3(\bar{L})$, we have $\bar{P} \cong 3^{1+2}$, so $\bar{Y}_1 = \bar{P}\bar{P}^{\bar{a}}$ is $\bar{a}$-invariant and isomorphic to 3^{1+4}. Hence $m([V,\bar{a}]) \geq 9$ by Lemma 3.2(v), contradiction. This proves (i).

Next, suppose that (ii) fails. Then there is a hyperplane $\bar{B}$ of $\bar{A}$ and a subgroup $\bar{Y}$ of $C_{Z(\bar{L})}(\bar{B})$ of prime order such that $\bar{Y} = [\bar{Y},\bar{A}]$. Now $\bar{A}\bar{Y}$ centralizes $\bar{B}$, so normalizes $C_V(\bar{B})$. But since $\bar{L} = F^*(\bar{X})$, our choice of $\bar{A}$ allows us to apply Lemma 3.1(v). If $\bar{B} \neq 1$, we conclude that $\bar{A}$ centralizes $C_V(\bar{B})$, and so $\bar{Y} = [\bar{Y},\bar{A}]$ does as well. Hence by the $A \times B$ lemma, $\bar{Y}$ centralizes V, a contradiction. Thus $\bar{B} = 1$, so by Lemmas 3.1(i) and 3.2(ii), $\bar{A} \leq O_{2'2}(\bar{X})$. Hence $[\bar{A},\bar{L}] = 1$, contradiction.

Thus (ii) holds, and since (iii) is immediate from Lemma 3.1(v), the lemma follows.

We can now prove

LEMMA 3.7: *If $\bar{L}$ is of Lie type of odd characteristic, then $\bar{L} \cong 3U_4(3)$ or $[3\times 3]U_4(3)$.*

Proof: If $|\bar{A}| > 2$, then because of the quadratic action of Lemma 3.6(iii), the lemma follows from Theorem 1.2. Thus we can assume $\bar{A} = \langle \bar{a} \rangle \cong Z_2$, so that $\bar{a}$ induces a transvection on V. We shall argue in this case that $\bar{X}$ contains a four-group acting quadratically on V, so that the desired conclusion will again follow from Theorem 1.2. By Glauberman's Z^*-theorem [Gl2] there is an $\bar{X}$-conjugate $\bar{b}$ of $\bar{a}$ such that $\bar{B} = \langle \bar{a},\bar{b} \rangle \cong E_4$. Since $\bar{b}$ is conjugate to $\bar{a}$, likewise $\bar{b}$ acts as a transvection on V. Set $V_1 = [V,\bar{a}]$ and $V_2 = [V,\bar{b}]$, so that $V_1 \cong V_2 \cong Z_2$ and $\bar{B} = \langle \bar{a},\bar{b} \rangle$ centralizes V/V_1V_2. Thus $[V,\bar{B}] \leq V_1V_2$. But also V_1 and V_2 are each $\bar{B}$-invariant as $\bar{B}$ is abelian. Since each has order 2, $\bar{B}$ centralizes both, so $[V,\bar{B},\bar{B}] = 1$. Thus $\bar{B}$ is a quadratic four-group in its action on V, and the lemma is proved.

We now eliminate this residual case (the argument is due to Aschbacher [As3; 10.7]).

LEMMA 3.8: *$\bar{L}$ is not of Lie type of odd characteristic.*

Proof: Suppose false and continue the preceding analysis. Then $\bar{L} \cong 3U_4(3)$ or $[3\times 3]U_4(3)$ and by the choice of A, $A \cap C$ has maximal order among all elements A of $\mathcal{A}(T)$. Set $\bar{Y} = Z(\bar{L})$, so that $[\bar{Y},\bar{A}] = 1$ by Lemma 3.5.

Thus $\bar{Y} \leq Z(\bar{X})$. If $\bar{Y} \cong E_9$, then for some $1 \neq \bar{y} \in \bar{Y}$, $V_0 = C_V(\bar{y}) \neq 1$ and $[\bar{Y}, V_0] \neq 1$. It follows that $V_0 \triangleleft X$ and $C_{\bar{X}}(V_0) = \langle \bar{y} \rangle$, so V_0 is 2-reduced and singular with $F^*(X/C_X(V_0)) \cong 3U_4(3)$. This contradicts the minimality of $\bar{X}$. Therefore, $\bar{L} \cong 3U_4(3)$. Consequently, since $\mathrm{Out}(U_4(3)) \cong D_8$ acts faithfully on the 3-part of the Schur multiplier of $U_4(3)$ (which is E_9) by [GL; 7–8], and $\bar{X} = \bar{L}\bar{A}$, we must have $|\bar{X} : \bar{L}| \leq 2$. This implies that $m(\bar{A}) \leq 1 + m_2(\bar{L})$, so $m(\bar{A}) \leq 5$.

Let W be a minimal $\bar{L}$-invariant subgroup of V on which $\bar{Y}$ acts nontrivially. Then $W = [W, \bar{Y}]$ and W may be viewed as an irreducible $\mathbf{F}_4\bar{L}$-module. Since $\bar{Y} \leq \bar{L}' \cap Z(\bar{L})$, $\dim_{\mathbf{F}_4}(W)$ is divisible by 3, and since $|\bar{L}| > |GL_3(4)|$, $\dim_{\mathbf{F}_4}(W) \geq 6$. For any $a \in A^{\#}$, $m(V/C_V(a)) \leq m(\bar{A}) \leq 5$ by Lemma 3.1, so $C_W(a) \neq 1$, and then $W = W^a$ by irreducibility of W, as $W \cap W^a$ is $\bar{L}$-invariant.

If $\bar{A} = \langle \bar{a} \rangle$, then $\bar{a}$ acts as a transvection on W, whence $[V, \bar{a}] \cong Z_2$. But $\bar{Y}$ leaves $[W, \bar{a}]$ invariant as $\bar{a}$ centralizes $\bar{Y}$, so $\bar{Y}$ centralizes $[W, \bar{a}]$. Thus $C_W(\bar{Y}) \neq 1$, contrary to the irreducible action of $\bar{L}$ on W. Hence $|\bar{A}| > 2$ and so $\bar{A} \cap \bar{L} \neq 1$. But $\bar{L}$ has only one class of involutions [Co]. Hence we can choose $\bar{a} \in \bar{A} \cap \bar{L}$ such that $\bar{a}$ normalizes a 3-subgroup $\bar{P}$ of $\bar{L}$ with $\bar{Y} \leq \bar{P}$ and $\bar{P}/\bar{Y} \cong 3^{1+4}$ and $\bar{a}$ inverting $\bar{P}/\bar{Y}/\Phi(\bar{P}/\bar{Y})$; namely, $\bar{P}/\bar{Y} = O_3(\bar{H})$ for an appropriate maximal parabolic subgroup $\bar{H}$ of $\bar{L}/\bar{Y}$.

Given the structure of $[\bar{P}, \bar{a}]$, Lemma 3.2(iv) yields now that $k = m([W, \bar{a}]) > 2$. However, as W is a vector space over $\mathbf{F}_4$, k must be even, whence $k \geq 4$. On the other hand, $m(\bar{A}) \leq 5$, so as $C_W(A) \leq C_W(\bar{a})$, $m(W/C_W(\bar{a})) = k \leq 5$ by Lemma 3.1, forcing $k = 4$. But likewise $m(C_W(A))$ is even, and again $m(W/C_W(A)) \leq m(\bar{A}) \leq 5$, so as $m(\bar{A}) \leq 5$, we conclude that $m(W/C_W(A)) = 4$, whence $C_W(A) = C_W(\bar{a})$ and then $m(\bar{A}) \geq 4$.

Finally, setting $\bar{B} = \bar{A} \cap \bar{L}$, we have $m(\bar{B}) \geq 3$ and as $\bar{a}$ was arbitrary in $\bar{A}^{\#} \cap \bar{L}$, it follows that $W_0 = C_W(\bar{A}) = C_W(\bar{b})$ for all $\bar{b} \in \bar{B}^{\#}$. Thus W_0 is invariant under each $C_{\bar{L}}(\bar{b})$. But by [Co], each $C_{\bar{L}}(\bar{b})$ is a maximal subgroup of $\bar{L}$, and $\bar{L}$ has only one class of involutions and Sylow 2-center of order 2. Thus, $C_{\bar{L}}(\bar{b}) \neq C_{\bar{L}}(\bar{b}')$ for $\bar{b} \neq \bar{b}' \in \bar{B}^{\#}$, and so $\bar{L} = \langle C_{\bar{L}}(\bar{b}) \mid \bar{b} \in \bar{B}^{\#} \rangle$. Therefore $\bar{L}$ leaves W_0 invariant. Since $1 \neq W_0 < W$, this contradicts the irreducibility of $\bar{L}$ on W. The lemma follows.

Similarly we prove (see [As3; §9])

LEMMA 3.9: $\bar{L} \not\cong 3A_7$.

Proof: Suppose that $\bar{L} \cong 3A_7$. By Lemma 3.6, $Z(\bar{L}) \leq Z(\bar{X})$. However, non-inner automorphisms of $\bar{L}$ invert $Z(\bar{L})$. (A Sylow 3-subgroup $\bar{D}$ of $\bar{L}$ is nonabelian, since otherwise by Burnside transfer, $Z(\bar{L}) \leq \bar{L}' \cap Z(N_{\bar{L}}(\bar{D})) = 1$. Thus $\bar{D}$ is isomorphic to 3^{1+2}. We may take $\bar{D}/Z(\bar{L}) = \langle(123),(456)\rangle$, and then $t = (12)$ normalizes $\bar{D}$ with $|C_{\bar{D}/Z(\bar{L})}(t)| = 3$, so t inverts $Z(\bar{L})$.) Therefore $\bar{X} = \bar{L}$. Choose $\bar{a} \in \bar{A}^{\#}$.

Since $\bar{L}$ has only one class of involutions, $\bar{a}$ normalizes a 3^{1+2}-subgroup $\bar{D}$ of $\bar{L}$, centralizing $Z(\bar{D})$ and inverting $\bar{D}/Z(\bar{D})$. Set $W = [V, Z(\bar{D})]$, so that $C_W(Z(\bar{D})) = 1$. Also W is $\bar{X}$-invariant. Hence by Lemma 3.2(iii), $m(W) = 6r$ and $m([V,\bar{a}]) \geq 2r$. It follows now from Lemma 3.1 that $m(\bar{A}) \geq 2r$. However, $m(\bar{A}) \leq m_2(\bar{L}) = 2$, so $r = 1$ and consequently $m(W) = 6$.

By [GL, 8-1], $C_{GL(W)}(Z(\bar{L})) \cong GL_3(4)$, so $\bar{L}$ is a subgroup of $GL_3(4)$ and hence of $SL_3(4)$. However, $|SL_3(4)| = 8|3A_7|$, so if $3A_7 \leq SL_3(4)$, it would follow that $L_3(4) \leq A_8$, so $L_3(4) \cong A_8$, which is not the case.

We next prove

LEMMA 3.10: *$\bar{L} \not\cong Sz(2^n)$ or $(S)U_3(2^n)$, $n \geq 2$.*

Proof: Suppose first that $\bar{L} \cong Sz(2^n)$. Then $\mathrm{Out}(\bar{L})$ has odd order by [GL; 7-1], so $m_2(\mathrm{Aut}(\bar{L})) = n$. Therefore $m(\bar{A}) \leq n$ and so $m([V,\bar{a}]) \leq n$ for any $\bar{a} \in \bar{A}^{\#}$ by Lemma 3.1. On the other hand, if p is a primitive prime divisor of $2^{2n}+1$ (which exists by Zsigmondy's theorem), then $\bar{L}$ has a strongly real element $\bar{x}$ of order p. Thus, since $\bar{L}$ has one class of involutions, we may take $\bar{x}$ to be inverted by $\bar{a}$, and thus $d(\bar{x}) \leq 2n$ by Lemma 3.2(i). However, by choice of p, $d(\bar{x}) \geq 4n$, contradiction.

If $\bar{L} \cong (S)U_3(2^n)$, then $\mathrm{Out}(\bar{L})$ has cyclic Sylow 2-subgroups by [GL; 7-1], and $m_2(\bar{L}) = n$, so $m(\bar{A}) \leq n+1$. If $\bar{A}$ contains an involution $\bar{a}$ inducing a non-inner automorphism on $\bar{L}$, then some conjugate of $\bar{a}$ is induced by an automorphism of $\mathbf{F}_{2^{2n}}$ of order 2, so $\bar{a}$ inverts an element $\bar{x}$ of order p, a primitive prime divisor of $2^{3n}+1$. Using Lemmas 3.1 and 3.2 as above, we get $m([V,\bar{a}]) \leq n+1$, so $d(\bar{x}) \leq 2(n+1)$, whereas $d(\bar{x}) = 6n$, contradiction. Thus $\bar{A}$ induces inner automorphisms on $\bar{L}$, so $m(\bar{A}) \leq n$.

Let $\bar{a} \in \bar{A}^{\#}$, so that $m([V,\bar{a}]) \leq n$. Suppose first that $n \neq 3$ and let r be a primitive prime divisor of 2^n+1. Then the monomial subgroup of $\bar{L}$ contains a subgroup $\bar{R}\bar{W}$, where $\bar{R} \cong E_{r^2}$, $\bar{W} \cong \Sigma_3$, and $\bar{W}$ acts faithfully on $\bar{R}$. Since $\bar{L}$ has one class of involutions, $\bar{R}\bar{W}$ is generated by three conjugates $\bar{a}_1, \bar{a}_2, \bar{a}_3$ of

$\bar{a}$, and so $|V : C_V(\bar{R}\bar{W})| \leq |V : \cap_{i=1}^{3} C_V(\bar{a}_i)| \leq |[V,\bar{a}]|^3 \leq 2^{3n}$. Hence $\bar{R}$, being of odd order, embeds in $GL_{3n}(2)$. But $GL_{3n}(2)$ has cyclic Sylow r-subgroups by [GL; 10.1], a contradiction. Thus $n = 3$.

A similar argument works in this case, with $\bar{R} \cong Z_9 \times Z_9$ or $Z_9 \times Z_3$, according as $Z(\bar{L}) \neq 1$ or $Z(\bar{L}) = 1$. This time, $\bar{R}\bar{W}$ is again generated by three conjugates of $\bar{a}$ and so $O_3(\bar{R}\bar{W})$ embeds in $GL_9(2)$. But $O_3(\bar{R}\bar{W})$ either contains $Z_9 \times Z_9$ or is of maximal class and order 3^4 with a $Z_9 \times Z_3$ subgroup. Since Sylow 3-subgroups of $GL_9(2)$ are $Z_3 \times (Z_3 \wr Z_3)$, we reach a contradiction in either case, and the lemma is proved.

It thus remains to eliminate the sporadic cases.

LEMMA 3.11: *$\bar{L} \not\cong J_1$ or Ly.*

Proof: Suppose false. Then $m(\bar{A}) \leq 4$, so $m([V,\bar{a}]) \leq 4$ for any $\bar{a} \in \bar{A}^{\#}$ by Lemma 3.1. But $\bar{L}$ has only one class of involutions and so $\bar{a}$ inverts an element $\bar{x}$ of $\bar{L}$ of order 11 or 37 by [Co]. Now $d(\bar{x}) \geq 10$, so $m([V,\bar{a}]) \geq 5$ by Lemma 3.2(i), contradiction.

This completes the proof of Proposition 3.5.

Next we analyze failure of factorization for $L_2(2^n)$ and A_{2n+1}. The conclusions are fundamental for the proof of Theorem A. Moreover, because $L_2(4) \cong A_5$, it is best to treat the linear and alternating cases together.

However, we first note that in [As3], Aschbacher analyzes the corresponding situation for alternating groups of arbitrary degree. His more general result makes for an easier induction argument, but to start the induction he requires knowledge of the basic $\mathbf{F}_2$-modules for $A_8(\cong L_4(2))$. By restricting ourselves to the odd degree case, we avoid the latter issue.

We shall prove

PROPOSITION 3.12: *Assume that $F^*(\bar{X}) = \bar{L} \cong L_2(2^n)$ or A_{2n+1}, $n \geq 2$. Set $U = [V,\bar{L}]$. Let $A \in \mathcal{A}(T)$ with $\bar{A} \neq 1$. Then $(A \cap C)U = (A \cap C)V \in \mathcal{A}(T)$ and one of the following holds:*

(i) *$\bar{L} \cong L_2(2^n)$, and*
   (1) *$U/C_U(\bar{L})$ is a natural $L_2(2^n)$-module;*
   (2) *$\bar{A} \in \mathrm{Syl}_2(\bar{L})$;*
   (3) *$m(U/C_U(\bar{A})) = n$;*
   (4) *$C_V(\bar{A})$ is in the center of a Sylow 2-subgroup of the preimage of $\bar{L}$ in X; and*

(5) *$\bar{S}$ induces inner automorphisms on $\bar{L}$;*

(ii) $\bar{L} \cong A_{2n+1}$, $\bar{L}\bar{S} \cong \Sigma_{2n+1}$, *and*

(1) *U is a natural $\bar{X}$-module;*

(2) $V = U \times C_V(\bar{L})$;

(3) *$\bar{X} = \bar{L}\bar{A}$ and $\bar{A}$ contains some $\bar{a}$ acting on $\bar{L}$ as a transposition;*

(4) $\bar{J} = \langle \bar{a}^{\bar{X}} \cap \bar{T} \rangle \cong E_{2^n}$;

(5) $|\bar{A}| = |V : C_V(\bar{A})|$;

(6) *If $\bar{A}$ is minimal (subject to $\bar{A} \neq 1$), then $|\bar{A}| = 2$; and*

(7) $C_{\bar{T}}(C_V(\bar{J})) = \bar{J}$; *or*

(iii) $\bar{L} = \bar{X} \cong A_7$, *and*

(1) $U \cong E_{16}$;

(2) $V = U \times C_V(\bar{L})$;

(3) *$\bar{A} = \bar{J} \cong E_4$ is a root four-subgroup of $\bar{L}$; and*

(4) *$\bar{L}$ permutes $U^\#$ transitively.*

Remark: Following this proposition, we shall deal with the ambiguity $L_2(4) \cong A_5$ as follows. If $\bar{L}$ satisfies the hypothesis of the proposition and $\bar{L} \cong A_5$, we write $\bar{L} \cong L_2(4)$ if conclusion (i) holds and $\bar{L} \cong A_5$ if conclusion (ii) holds. ∎

Proof: We shall consider three cases and show that they lead to conclusions (i), (ii), (iii), respectively.

Case 1. $\bar{L} \cong L_2(2^n)$, $n \geq 2$; and if $n = 2$, then $\bar{A} \leq \bar{L}$.

First assume that $\bar{A} \cap \bar{L} = 1$. Then $n \geq 3$ by assumption, and $\bar{A} = \langle \bar{a} \rangle$ with $\bar{a}$ inducing a field automorphism on $\bar{L}$. Hence $\bar{a}$ inverts a subgroup of a Cartan subgroup of order $2^{n/2}+1$. By Lemma 3.2, $|\bar{A}| = 2$ forces $2^{n/2}+1 = 3$, so $n = 2$, contradiction. Thus $\bar{A} \cap \bar{L} \neq 1$.

For any $\bar{a} \in \bar{A} \cap \bar{L}^\#$, $\bar{a}$ inverts some $\bar{x} \in \bar{L}$ of order $2^n + 1$. But $d(\bar{x}) \geq 2n$ by Zsigmondy's theorem, and so $m([U, \bar{a}]) \geq n$ by Lemma 3.2. Since $A \in \mathcal{A}(T)$, it follows that $m(\bar{A}) \geq m(V/C_V(A)) \geq m(U/C_U(A)) \geq m(U/C_U(\bar{a})) = m([U,\bar{a}]) \geq n$. This forces $\bar{A} \in \mathrm{Syl}_2(\bar{L})$, so equality holds throughout, and hence $m(\bar{A}) = m(V/C_V(A)) = m(U/C_U(A))$. Now (i2-i4) follow, and also $(A \cap C)U \in \mathcal{A}(T)$ (see Lemma 3.1(ii)). Furthermore, $C_V(\bar{A}) = C_V(\bar{a})$ for all $\bar{a} \in \bar{A}^\#$.

Let $\bar{B}$ be a subgroup of $\bar{A}$ of order 4. Since $C_V(\bar{A}) = C_V(\bar{a})$ for all $\bar{a} \in \bar{A}^\#$, V has no free summand, as $\bar{B}$-module. Let W be a nontrivial $\bar{L}$-composition factor in V; then W has no free $\bar{B}$-summand. The module W, with scalars extended to the algebraic closure $\bar{\mathbf{F}}_2$ of $\mathbf{F}_2$, is then a direct sum of nontrivial (absolutely) irreducible $\bar{L}$-modules with no free $\bar{B}$-summand. But by the Steinberg tensor product theorem [S; §12], every such irreducible module has the form $M = N^{\sigma_1} \otimes \cdots \otimes N^{\sigma_r}$ for some $r \geq 1$, where N is a natural $\bar{L}$-module and $\sigma_1, \ldots, \sigma_r$ are distinct automorphisms of $\mathbf{F}_{2^n}$. It follows that $r = 1$, for otherwise we can take $\bar{B}$ to be generated by root elements $x(1)$ and $x(t)$, where $t^{\sigma_1} \neq t^{\sigma_2}$, and calculate immediately that M has a free $\bar{B}$-summand. We conclude that M is a natural module. Hence also W is. In particular, $m(W/C_W(\bar{A})) = n$, so $m(W/C_W(\bar{A})) = m(V/C_V(\bar{A}))$. Thus W is unique, and (i1) is immediate.

Finally, by (2), $\bar{J} = \bar{A}$, and so $m(U/C_U(\bar{J})) = n$. Since $U \leq V \leq Z(C)$, $C_U(\bar{J}) \leq Z(J)$. Thus S centralizes $C_U(\bar{J})$. But if $\bar{S} > \bar{J}$, then $\bar{S}$ would contain some $\bar{y}$ acting on $\bar{L}$ as a field automorphism. Then $\bar{y}$ also acts on the natural module $U/C_U(\bar{L})$ as a field automorphism, so acts freely on $C_U(\bar{A}) = C_U(\bar{J})$. This is a contradiction, so (i5) holds, thus completing the proof in case 1.

It therefore remains to consider the cases $\bar{L} \cong A_{2n+1}$, $n \geq 3$, and $\bar{X} = \bar{L}\bar{A} \cong \Sigma_5$, which we treat simultaneously by induction on n. In either case, $\bar{X} \leq \Sigma_{2n+1}$. This time we choose $A \in \mathcal{A}(T)$ with $\bar{A} \neq 1$ and $|\bar{A}|$ minimal. According as $|\bar{A}| = 2$ or $|\bar{A}| > 2$, we shall argue that the conclusions of (ii) or (iii) hold.

Case 2. $|\bar{A}| = 2$.

First, write $\bar{A} = \langle \bar{a} \rangle$, so that by Lemma 3.2, $\bar{a}$ induces a transvection on V. If $\bar{a}$ is the product of k disjoint transpositions, then $\bar{a}$ inverts a $(2k+1)$-cycle $\bar{x}$ in $\bar{L}$, so by Lemma 3.2, $d(\bar{x}) \leq 2$, whence $k = 1$; i.e., $\bar{a}$ is a transposition. In particular $\bar{X} \cong \Sigma_{2n+1}$.

Identify $\bar{X}$ with Σ_Ω, where $\Omega = \{1, 2, \ldots, 2n+1\}$, and assume that $\bar{a} = (12)$. For any $\Psi \subset \Omega$, write Σ_Ψ (respectively, A_Ψ) for the pointwise stabilizer of $\Omega - \Psi$ in Σ_Ω (respectively, A_Ω). If $\{1,2\} \subset \Psi \subset \Omega$ and $|\Psi|$ is odd, then since $\bar{A} \leq \Sigma_\Psi$, induction applied to the preimage of Σ_Ψ gives that $V = [A_\Psi, V] \times C_V(A_\Psi)$ and $[A_\Psi, V]$ is a natural A_Ψ-module. By conjugation, the same conclusion then holds for any $\Psi \subset \Omega$ with $|\Psi|$ odd.

Let W be a nontrivial X-chief factor in V. We use the preceding conclusion to prove that $\dim(W) \geq 2n$. Indeed, if $n = 2$, then this is clear as $|A_5|$ is divisible by 5. For larger n, write $\Omega = \Psi_1 \cup \Psi_2$ with $|\Psi_1| = 4$ and $|\Psi_2| = 2n - 3 \geq 3$. Let $Y = O_2(A_{\Psi_1}) \cong E_4$. By embedding A_{Ψ_1} in $A_{\Psi_1 \cup \{\omega\}}$ for some $\omega \in \Psi_2$ and applying the previous paragraph, we see that $W = W_1 \times W_2$, where Y is free on $W_1 \cong E_{16}$ and Y centralizes W_2. By the $A \times B$ lemma A_{Ψ_2} acts faithfully on

$C_W(Y) = C_{W_1}(Y) \times W_2$. But $C_{W_1}(Y) = C_W(Y) \cap [W,Y]$ is A_{Ψ_2}-invariant, and of order 2, so A_{Ψ_2} acts faithfully on $C_W(Y)/C_{W_1}(Y)$. By the previous paragraph this module contains a copy of the natural A_{Ψ_2}-module. Hence $\dim(W_2) = \dim(C_W(Y)/C_{W_1}(Y)) \geq 2n-4$, so $\dim(W) = \dim(W_1) + \dim(W_2) \geq 2n$, as claimed.

On the other hand, there exist $2n$ conjugates $\bar{a} = \bar{a}_1, \ldots, \bar{a}_{2n}$ of $\bar{a}$ which generate $\bar{X}$, and such that $\langle \bar{a}_1, \ldots, \bar{a}_{2n-1} \rangle = \Sigma_{\Omega - \{\omega\}}$ for some $\omega \in \Omega$. As each $\bar{a}_i$ is a transvection on V, $\dim(C_V(\Sigma_{\Omega-\{\omega\}})) \geq \dim(V) - (2n-1) > 0$. Thus $\Sigma_{\Omega-\{\omega\}}$ fixes some $w \in W^{\#}$. Therefore $\bar{w}$ has $2n+1 = |\Sigma_\Omega : \Sigma_{\Omega-\{\omega\}}|$ conjugates under $\bar{X}$, and these are then permuted naturally by $\bar{X}$, so W is a homomorphic image of the permutation module, and hence is a natural module.

To establish (1) and (2), it now suffices to show that $H^1(\bar{X}, W)$ is trivial. For then, since $\bar{a}$ is a transvection, W is the only nontrivial $\bar{X}$-chief factor in V, and by the trivial cohomology and the fact that W is self-dual, W has no nontrivial extensions by trivial modules, so it splits in V, yielding (1) and (2). To prove that $H^1(\bar{X}, W)$ is trivial, in turn, we consider an $\mathbf{F}_2\bar{L}$-module W^* containing W with codimension 1. There exist 3-cycles $\bar{t}_1, \ldots, \bar{t}_n$ which generate $\bar{L}$ (each moving two points fixed by all the preceding). We have $[W^*, \bar{t}_i] = [W, \bar{t}_i] \cong E_4$, so $m(W^*/C_{W^*}(\bar{L})) \leq \sum_{i=1}^n m(W^*/C_{W^*}(\bar{t}_i)) = 2n$. Hence $C_{W^*}(\bar{L}) \neq 1$, so W^* splits over W, the desired conclusion. Thus (ii1) and (ii2) hold.

Next, consider an arbitrary $B \in \mathcal{A}(T)$ with $\bar{B} \neq 1$. If $\bar{B}$ has r orbits on Ω, then $\dim(C_U(\bar{B})) = r-1$ since U is a natural $\bar{X}$-module. Since $B \in \mathcal{A}(T)$, $m(\bar{B}) \geq m(V/C_V(\bar{B})) = m(U/C_U(\bar{B})) = 2n - (r-1) = (2n+1) - r$. Thus $|\Omega| \leq m(\bar{B}) + \ell$, where ℓ is the number of orbits of $\bar{B}$. But an orbit of $\bar{B}$ has order 2^a and the corresponding action group of $\bar{B}$ has rank a for some a. Since the rank of $\bar{B}$ is at most the sum of the ranks of the action groups, it follows that each $a \leq 1$ and hence that $\bar{B}$ is generated by disjoint transpositions and equality holds, whence $m(\bar{B}) = m(V/C_V(\bar{B}))$. This yields (ii3) and (ii5). Since the transpositions in $\bar{T}$ are pairwise disjoint and are permuted transitively by the normalizer of the group they generate, (ii4) follows as well. Furthermore, if $b \in B$ and $\bar{b}$ is a transposition, then $\bar{b}$ is a transvection on V, and $|V : C_V(\bar{b})| = |\bar{b}|$. Hence $|C_V(\bar{b}) : C_V(\bar{B})| = |\bar{B} : \langle \bar{b} \rangle|$, so $\langle b \rangle (B \cap C) C_V(\bar{b}) \in \mathcal{A}(T)$, proving (ii6).

Finally, since $U \times Z_2$ is the permutation module (of dimension $2n+1$) for $\bar{X}$, it is clear that for any $\bar{H} \leq \bar{X}$, $C_{\bar{X}}(C_U(\bar{H}))$ is the intersection of the setwise stabilizers of the orbits of $\bar{H}$. Since $\bar{J}$ is generated by n disjoint transpositions, (ii7) follows, completing the proof in this case.

Case 3. $|\bar{A}| > 2$.

Suppose first that there is $\Psi \subset \Omega$ with $|\Psi|$ odd, Ψ invariant under $\bar{A}$ and not fixed pointwise by $\bar{A}$. We shall prove that

$$(3.2)\qquad |\Psi| = 5 \text{ or } 7, \bar{A} \cong E_4, \text{ and } \bar{A} \text{ acts as a root four-group on } \Psi \text{ (i.e., has exactly one nontrivial orbit on } \Psi).$$

Indeed, let $\bar{Y} = \Sigma_\Psi \subset \Sigma_\Omega = \bar{X}$, so that $\bar{Y}$ is $\bar{A}$-invariant. Let $\bar{X}_1 = \bar{Y}\bar{A}$ and let X_1 be the preimage of $\bar{X}_1$ in X. Setting $V_1 = C_V(O_2(\bar{X}_1))$ and $\tilde{X}_1 = X_1/C_{X_1}(V_1)$, we have by Lemma 3.3 that V_1 is 2-reduced in X_1 and $F^*(\tilde{X}_1) = O^2(\tilde{Y}) \cong A_\Psi$. By induction, the lemma holds for X_1 and V_1. In particular, $A^* = (A \cap C_{X_1}(V_1))V_1 \in \mathcal{A}(T)$. By the minimality of $\bar{A}$, $\overline{A^*} = 1$ or $\bar{A}$. But $[\bar{A}, V_1] \neq 1$, since otherwise $O^2(\bar{Y}) = [\bar{Y}, \bar{A}]$ would centralize V_1, whence $[O^2(\bar{Y}), V] = 1$ by the $A \times B$ lemma, contradicting $O^2(\bar{Y}) \neq 1$. Thus $\overline{A^*} \neq \bar{A}$, so $\overline{A^*} = 1$. But $C_{\bar{X}_1}(V_1) \geq O_2(\bar{X}_1)$, so $\bar{A} \cap O_2(\bar{X}_1) = 1$; i.e., $\bar{A}$ acts faithfully on $\bar{Y}$. Furthermore, if (ii) holds for X_1, then by (ii6) and the minimality of $|\bar{A}|$, $|\bar{A}| = 2$, contradiction. Thus (i) or (iii) holds for X_1, whence $|\Psi| = 5$ or 7 and $\bar{A}$ acts as a root four-group on $\bar{Y}$. This establishes (3.2). In particular, $\bar{A}$ has no orbit of length 2.

We argue next that

$$(3.3)\qquad \text{Either } 2n + 1 = 2^m + 1 \text{ for some } m \text{ or } 2n + 1 = 7.$$

Suppose false, so that $2n + 1 \geq 11$ and $2n + 1 \neq 2^m + 1$. The orbits of $\bar{A}$ are of sizes 1 and 2^a, for various $2^a \leq 2n$. There thus exists a Ψ satisfying the above condition with $|\Psi| = 2^a + 1$ for some such $a \geq 1$. The preceding analysis therefore yields that $|\bar{A}| = 4$ and $\bar{A}$ has no orbit of length 2. But then since $2n + 1 \geq 11$, $\bar{A}$ stabilizes and acts nontrivially on some $\Psi \subset \Omega$ with $|\Psi| = 9$, contradicting the previous paragraph. This proves (3.3).

If $2n + 1 = 5$, then by assumption $\bar{X} = \bar{L}\bar{A} \cong \Sigma_5$. Hence $\bar{A}$ has an orbit of length 2, a contradiction.

If $2n + 1 = 7$, we argue that (iii) holds. Since $\bar{A}$ has no orbit of length 2, it is a root four-group; in particular, $\bar{A} \leq \bar{L}$. By [GL; 23-1], $\langle \bar{A}, \bar{A}^{\bar{x}} \rangle = \bar{L}$ for some $\bar{x} \in \bar{L}$, and so $\bar{L}$ centralizes $C_V(\bar{A}) \cap C_V(\bar{A})^{\bar{x}}$, of index ≤ 16 in V. Thus in V there is a unique nontrivial $\bar{L}$-composition factor W, and $|W| = 16$. But $H^1(\bar{X}, U)$ is trivial for $U = W$ and its dual. (If $\bar{N} = N_{\bar{X}}(\bar{R})$, $\bar{R} \leq \bar{X}$, $\bar{R}$ generated by a

3-cycle, then $C_U(\bar{R}) = 1$, so $H^1(\bar{N}, U)$ is trivial; since $|\bar{X} : \bar{N}|$ is odd, $H^1(\bar{X}, U)$ is then trivial.) Therefore $V = W \times C_V(\bar{L})$. Then $\bar{X}$ acts faithfully on W, and since $L_4(2) \cong A_8$ contains no copy of Σ_7, $\bar{X} = \bar{L}$. Also as $3^2 \cdot 5$ divides $|\bar{L}|$, $\bar{L}$ is transitive on $W^{\#}$. In addition, $\bar{A}$ is uniquely determined in $\bar{T}$ as its only root four-subgroup, so $\bar{A} = \bar{J}$. Since $|W| = 16$, it follows that $|V : C_V(\bar{A})| = 4 = |\bar{A}|$. Thus all parts of (iii) hold.

We are left to treat the case $2n+1 = 2^m + 1$, $m \geq 3$, and proceed to derive a contradiction. Notice that every nontrivial $\bar{L}$-composition factor W in V has rank greater than $2m$. (If $m = 3$, $\bar{L} \cong A_9$ contains a Frobenius group of order $9 \cdot 8$, so $m_2(W) \geq 8$; if $m > 3$, $\bar{L}$ contains $(Z_3)^r$, where $r = [(2^m+1)/3] > m$, so $m_2(W) \geq 2r > 2m$.)

We claim that

$$\text{Either } m(\bar{A}) = 2 \text{ or } m;$$

(3.4) $$\text{Every orbit of } \bar{A} \text{ on } \Omega \text{ is trivial or regular; and}$$

$$\text{If } m(\bar{A}) = 2, \text{ then } \bar{L} \cong A_9.$$

Indeed, if $\bar{A}$ has an orbit of length 2^m, (3.4) is obvious. In the contrary case, we apply (3.2) with Ψ equal to the union of a fixed point of $\bar{A}$ and a nontrivial $\bar{A}$-orbit to conclude that $\bar{A} \cong E_4$ and $\bar{A}$ acts as a root four-group on Ψ. Thus $|\Psi| = 5$. But then if $m > 3$, we can apply (3.2) to $\Psi_1 \subset \Omega$ of size 9 to derive a contradiction. Thus $m = 3$ and $2n + 1 = 9$. Furthermore, the same argument shows that $\bar{A}$ acts as a root four-group (and hence regularly) on each of its nontrivial orbits. Thus all parts of (3.4) hold.

Since $m(\bar{A}) \leq m$ and $m(W) > 2m$, as shown above, $\bar{L} \not\leq \langle \bar{A}, \bar{A}^{\bar{x}} \rangle$ for any $\bar{x} \in \bar{X}$ as $|V : C_V(\bar{A})| \leq 2^m$. However, if $|\bar{A}| = 2^m$, there is $\bar{x} \in \bar{L}$ such that $\langle \bar{A}, \bar{A}^{\bar{x}} \rangle = \bar{L}$, which is a contradiction.

To see that such an $\bar{x}$ exists, choose $\bar{a} \in \bar{A}^{\#}$ and let $\Omega_0 = \{\omega\}, \Omega_1, \ldots, \Omega_s$ be the orbits of $\bar{a}$ on Ω (so that $s = 2^{m-1}$ and $|\Omega_i| = 2$ for all $i \geq 1$). Let $\bar{x}_1$ be a 3-cycle supported on $\Omega_0 \cup \Omega_1$. Then $\bar{a}$ inverts $\bar{x}_1$, so $\langle \bar{A}, \bar{A}^{\bar{x}_1} \rangle \geq \langle \bar{x}_1, \bar{A} \rangle$. But $\bar{A}$ permutes the set of $\Omega_i (1 \leq i \leq s)$ transitively and so some $\bar{A}$-conjugate of $\bar{x}_1$ is a 3-cycle $\bar{x}_i$ supported on $\Omega_0 \cup \Omega_i$. Then $\langle \bar{A}, \bar{A}^{\bar{x}_1} \rangle \geq \langle \bar{x}_1, \ldots, \bar{x}_s \rangle = \bar{L}$, as asserted.

We conclude that $|\bar{A}| = 4$ and $\bar{L} \cong A_9$. In this case we can find $\bar{x}_1, \bar{x}_2 \in \bar{L}$ such that $\langle \bar{A}, \bar{A}^{\bar{x}_1}, \bar{A}^{\bar{x}_2} \rangle = \bar{L}$. (If $\bar{A}$ is a root four-group supported on Ω_1, choose the $\bar{x}_i$ so that $\Omega_1 \cap \Omega_1^{\bar{x}_i}$ has one element and $\Omega = \Omega_1 \cup \Omega_1^{\bar{x}_1} \cup \Omega_1^{\bar{x}_2}$. If $\bar{A}$ has orbits $\Omega_0 \cup \Omega_1 \cup \Omega_2$ with $|\Omega_1| = |\Omega_2| = 4$, choose $\bar{x}_i$ so that $\Omega_i \cup \Omega_i^{\bar{x}_i} = \Omega_0 \cup \Omega_i$.) Thus $\bar{L}$

centralizes $C_V(\bar{A}) \cap C_V(\bar{A})^{\bar{x}_1} \cap C_V(\bar{A})^{\bar{x}_2}$. But $|V : C_V(\bar{A})| \leq 4$, so any nontrivial $\bar{L}$-composition factor in V has order $\leq 2^6$, again a contradiction. Thus (iii) holds in case 3, and the proposition is proved.

Finally we prove

PROPOSITION 3.13: *If $F^*(\bar{X}) = \bar{L}$, and $\bar{L}/Z(\bar{L}) \cong L_3(2^n)$ or $Sp_4(2^n)'$, then no element of T maps to a graph or graph-field automorphism in $\mathrm{Out}(\bar{L})$; that is, the image of T in $\mathrm{Out}(\bar{L})$ acts trivially on the Dynkin diagram of $\bar{L}$, or equivalently, T normalizes all parabolic subgroups of $\bar{L}$ containing $\bar{T} \cap \bar{L}$..*

We assume false and let $t \in T$ induce such an automorphism on $\bar{L}$. We first prove

LEMMA 3.14: *$t \notin J$.*

Proof: If false, we could take $t \in A$ for some $A \in \mathcal{A}(T)$. Choose such an A with $|\bar{A}|$ minimal. Set $\bar{L}_1 = O^3(C_{\bar{L}}(\bar{t}))$, $\bar{X}_1 = \bar{L}_1\bar{A}$, and let X_1 be the preimage of $\bar{X}_1$ in X, $V_1 = C_V(O_2(\bar{X}_1))$, $C_1 = C_{X_1}(V_1)$, and $\tilde{X}_1 = X_1/C_1$. As usual, V_1 is 2-reduced in X_1. If $\bar{L} \cong (S)L_3(2^n)$, then $\bar{L}_1 \cong L_2(2^n)$ or $(S)U_3(2^{n/2})$, while if $\bar{L}/Z(\bar{L}) \cong Sp_4(2^n)'$, then $\bar{L}_1 \cong Sz(2^n)$ (or D_{10} if $n = 1$). (Given the structure of $\mathrm{Out}(\bar{L})$ [GL, 7-1], one can obtain a list of involutions in $\mathrm{Aut}(\bar{L}) - \mathrm{Inn}(\bar{L})$ and their centralizers by Lang's theorem and simple calculations; see [AS], for example.) In any case, $C_{\mathrm{Aut}(\bar{L})}(\bar{L}_1) \cong Z_2$, so $C_{\bar{X}_1}(\bar{L}_1) = \langle\bar{t}\rangle(Z(\bar{L}) \cap \bar{X}_1)$. Clearly $F^*(\tilde{X}_1) = F^*(\tilde{L}_1)$.

If $\tilde{A} \neq 1$, we conclude by Propositions 3.12 and 3.5(ii) that $A^* = (A \cap C_1)V_1 \in \mathcal{A}(T)$. Since $t \in A^*$, and $\overline{A^*} \leq \bar{A}$, our choice of A implies that $\bar{A} = \overline{A^*}$. But $\tilde{A}^* = 1$, so $\tilde{A} = 1$, contradiction. Thus, $\tilde{A} = 1$, so $\bar{A} = \langle\bar{t}\rangle$. Then by Lemma 3.2, every element $\bar{x}$ of $\bar{L}^{\#}$ of odd order inverted by $\bar{t}$ has order 3. However, $\bar{t}$ inverts an element of odd prime order r, where (1) if $\bar{L}_1 \cong L_2(2^n)$, r divides $2^{3n} - 1$, but not $|L_2(2^n)|$; (2) if $\bar{L}_1 \cong U_3(2^{n/2})$, r divides $2^{3n/2} - 1$, but not $2^n - 1$; (3) if $\bar{L}/Z(\bar{L}) \cong Sp_4(2^n)'$, $r = 5$. This contradiction establishes the lemma.

Similarly we prove

LEMMA 3.15: *If $A \in \mathcal{A}(T)$ and $a \in A$, then a does not induce a nontrivial field automorphism on $\bar{L}$.*

Proof: Indeed, if false, then since $\bar{a}$ acts freely on the lower central factors of $\bar{T}$, we could take $\bar{t}$ to centralize $\bar{a}$ by a Frattini-type argument. We form

$\bar{X}_1 = C_{\bar{L}}(\bar{a})\langle\bar{t},\bar{a}\rangle$ and apply induction to the preimage X_1 of $\bar{X}_1$ in X and to $V_1 = C_V(O_2(\bar{X}_1))$. Now $\bar{t}$ induces a graph automorphism on $C_{\bar{L}}(\bar{a})$, so we reach a contradiction by induction unless $\bar{A} = \langle\bar{a}\rangle$. In that case we again easily find an element $\bar{x}$ of $\bar{L}$ of odd order > 3 inverted by $\bar{a}$, contradiction.

We shall reduce to the case $\bar{L}/Z(\bar{L}) \cong Sp_4(2)'$. Thus in Lemmas 3.16–3.18, we assume

$$\bar{L}/Z(\bar{L}) \not\cong Sp_4(2)'. \tag{3.5}$$

Then the preceding analysis yields that $\bar{J} \leq \bar{L}$. We let $\bar{P}_1$ and $\bar{P}_2$ be the maximal parabolic subgroups of $\bar{L}$ containing $\bar{T} \cap \bar{L}$, set $\bar{Q}_i = O_2(\bar{P}_i)$, and let $\bar{L}_i = \bar{K}_i'$, where $\bar{K}_i$ is a Levi factor of $\bar{P}_i$, $i = 1,2$. Each $\bar{L}_i \cong L_2(2^n)$, and $\bar{J} \leq \bar{P}_i$.

We immediately obtain

LEMMA 3.16: *$\bar{J} \not\leq \bar{Q}_i$ for both $i = 1$ and 2.*

Proof: Suppose $\bar{J} \leq \bar{Q}_i$ for some i. Then $\bar{J} = \bar{J}^{\bar{t}} \leq \bar{Q}_i \cap \bar{Q}_i^{\bar{t}} = \bar{Q}_1 \cap \bar{Q}_2$. But then if Q_i is the preimage of $\bar{Q}_i$ in T, $J = J(Q_i)$ and so by the Frattini argument $\bar{J} \triangleleft \langle N_{\bar{X}}(\bar{Q}_1), N_{\bar{X}}(\bar{Q}_2)\rangle$, so $\bar{L}$ normalizes $\bar{J}$. As $\bar{L} = F^*(\bar{X})$ is simple and $[\bar{J},\bar{L}] \neq 1$, this is a contradiction.

Next, set $V_i = C_V(\bar{Q}_i)$, so that $\bar{L}_i$ acts on V_i, $i = 1,2$. We prove

LEMMA 3.17: *Either $\bar{L}_i$ acts trivially on V_i or $\bar{L}_i$ has a unique nontrivial composition factor on V_i, which is a natural $\bar{L}_i$-module, $i = 1,2$ (if $\bar{L}_i \cong Z_3$ this means that $[V_i,\bar{L}_i] \cong E_4$).*

Proof: We have $C_{\bar{P}_i}(\bar{L}_i\bar{Q}_i/\bar{Q}_i) \leq O_{22'}(\bar{P}_i)$, so $\bar{J}$ acts nontrivially on $\bar{L}_i\bar{Q}_i/\bar{Q}_i$, $i = 1,2$. But $C_X(V_i)$ contains C and covers $\bar{Q}_i$. If it covers $\bar{L}_i\bar{Q}_i$, then $\bar{L}_i$ acts trivially on V_i and the lemma holds. Otherwise, since $\bar{L}_i$ is simple and $C_{\bar{P}_i}(\bar{L}_i\bar{Q}_i/\bar{Q}_i)/\bar{Q}_i$ has odd order, we have $O_2(\bar{P}_i/C_{\bar{P}_i}(V_i)) = 1$; i.e., V_i is 2-reduced in the preimage of $\bar{P}_i$ in X. Moreover, V_i is singular since $[\bar{J},\bar{L}_i] \not\leq \bar{Q}_i$. Hence in this case the lemma follows from Proposition 3.12(i) as $\bar{J} \leq \bar{P}_i$.

We now contradict (3.5), proving

LEMMA 3.18: $\bar{L}/Z(\bar{L}) \cong Sp_4(2)'$.

Proof: Suppose false, so that $\bar{L} \cong (S)L_3(2^n)$, $n \geq 1$, or $Sp_4(2^n)$, $n \geq 2$. Set $V_0 = C_V(\bar{L})$ and let $\hat{W} = W/V_0$ be an $\bar{L}$-composition factor in V. Since $\bar{L}$ is perfect, $\hat{W}$ is not a trivial $\bar{L}$-module. Set $\hat{W}_i = C_{\hat{W}}(\bar{Q}_i) = W_i/V_0$, $i = 1,2$. Then

(3.6) Every $\bar{L}_i$-composition factor in $\hat{W}_i$ is trivial or a natural $\bar{L}_i$-module.

Indeed, setting $Y_i = C_W(\bar{Q}_i)$, we have just seen this for composition factors in $\hat{Y}_i$. Furthermore, since $[\bar{Q}_i, W_i] \le V_0$ and $[\bar{Q}_i, V_0] = 1$ by construction, commutation gives a pairing $\bar{Q}_i \times W_i \to V_0$; as $Y_i = C_W(\bar{Q}_i)$, this gives an embedding of W_i/Y_i into $\mathrm{Hom}(\bar{Q}_i, V_0)$, preserving the action of $\bar{L}_i$. Since all $\bar{L}_i$-composition factors of $\bar{Q}_i$ are trivial or natural, the same holds for W_i/Y_i. Thus (3.6) holds.

We shall argue on the basis of (3.6) and the existence of $\bar{a}$ that if we put $\hat{W}_1 = \hat{W} + \hat{W}^{\bar{a}}$, then

(3.7) $m(\hat{W}_1/C_{\hat{W}_1}(\bar{A})) > 2n$ or $3n$, according as $\bar{L} \cong (S)L_3(2^n)$ or $Sp_4(2^n)$.

Since $\bar{A} \le \bar{L}$, however, $m(\bar{A}) \le m(\bar{L}) = 2n$ or $3n$. Thus $m(V/C_V(\bar{A})) \ge m(\hat{W}_1/C_{\hat{W}_1}(\bar{A})) > m(\bar{A})$, which will contradict $A \in \mathcal{A}(T)$.

To see that (3.6) implies (3.7), a purely module-theoretic assertion about $\bar{L}$, we temporarily replace $\bar{L}$ for convenience by its simply connected version and first recall the representation theory of $\bar{L}$ over the algebraic closure $\bar{\mathbf{F}}_2$ [S; §12]. There are four basic $\bar{\mathbf{F}}_2\bar{L}$-modules, and the irreducible $\bar{\mathbf{F}}_2\bar{L}$-modules are the modules $\otimes_i M_i^{\sigma_i}$, the tensor product over $\mathrm{Gal}(\mathbf{F}_{2^n}/\mathbf{F}_2)$, with each M_i a basic module. Furthermore, three of the basic modules are (a) the trivial module, (b) M (a natural module), and (c) M' (an appropriate conjugate of M under a graph automorphism of $\bar{L}$), since the high weights for these modules lie in the restricted range. Let $N = M \otimes M'$. In the $L_3(2^n)$ case, M and M' are dual, and so $N \cong \mathrm{End}_{\bar{\mathbf{F}}_2}(M) = N_0 \oplus N_1$, where N_0 is the adjoint module (of dimension 8) and N_1 is trivial. In the $Sp_4(2^n)$ case, N is irreducible (indeed, its restriction to $Sp_4(2)$ is the Steinberg module) and we set $N_0 = N$. In either case, one calculates that the high weight of N_0 lies in the restricted range, so N_0 is the fourth basic module.

For any $\bar{\mathbf{F}}_2\bar{L}$-module U, write $U^{(i)}$ for the $\bar{\mathbf{F}}_2\bar{L}_i$-module $C_U(\bar{Q}_i)$, $i = 1, 2$. Clearly $(U_1 \otimes U_2)^{(i)}$ contains $U_1^{(i)} \otimes U_2^{(i)}$. We have

(3.8) If every irreducible constituent of $U^{(i)}$ is trivial or natural, for both $i = 1$ and 2, then every composition factor of U is trivial, or an algebraic conjugate of M, M', N_0, or $M^\sigma \otimes M'$ for some $\sigma \in \mathrm{Gal}(\bar{\mathbf{F}}_2)^\#$.

Indeed, clearly $M^{(i)}$ is natural for one i, and trivial for the other, and vice versa for $(M')^{(i)}$. Then since $N^{(i)} \supset M^{(i)} \otimes (M')^{(i)}$, $N^{(i)}$ and hence $N_0^{(i)}$ contains a natural module for both $i = 1$ and $i = 2$. Hence if $U_1 = \otimes_j (M_j)^{\sigma_j}$ is a composition factor of U, then except for the cases of (3.8), $U_1^{(i)}$ will contain, for some i, a submodule $\otimes_j(Y_j)^{\sigma_i}$, where each Y_j is natural or trivial and at least two Y_j are natural. By the Steinberg tensor product theorem again, this submodule is irreducible, but not a natural module, contrary to the assumption of (3.6). This establishes (3.8).

Now we prove (3.7). Let U be the module $\hat{W}$ with scalars extended to $\bar{\mathbf{F}}_2$. Because of (3.6), U satisfies the conclusion of (3.8). On the other hand, since $\hat{W}$ is an irreducible $\mathbf{F}_2\bar{L}$-module, U contains the direct sum of the distinct algebraic conjugates of an irreducible U_1. In each case of (3.8), U_1 clearly has exactly n such conjugates.

If $U_1 = N_0$ or $M^\sigma \otimes M'$, then for any involution $\bar{b} \in \bar{A}^\#$, $\dim(U_1/C_{U_1}(\bar{b})) \geq 3$ or 5 (according as $\bar{L} = (S)L_3(2^n)$ or $Sp_4(2^n)$) by a direct calculation of Jordan block structure of the tensor product of two involutions. Hence $m(\hat{W}/C_{\hat{W}}(\bar{b})) \geq 3n$ or $5n$, so (3.7) holds in this case.

Otherwise, we may assume without loss that $U_1 = M$. Thus $\hat{W} \not\cong \hat{W}^{\bar{a}}$, so $\hat{W}_1 = \hat{W} \oplus \hat{W}^{\bar{a}}$, and the extension of $\hat{W}^{\bar{a}}$ to $\bar{\mathbf{F}}_2$ is the direct sum of the n conjugates of $U_1' = M'$. For any involution $\bar{b} \in \bar{L}$, either $\dim(U_1/C_{U_1}(\bar{b}))$ or $\dim(U_1'/C_{U_1'}(\bar{b}))$ is at least 2 in the case of $Sp_4(2^n)$, and both are of course positive in any case. Hence $m(\hat{W}_1/C_{\hat{W}_1}(\bar{b})) \geq 2n$ or $3n$ according to the type of $\bar{L}$, so $m(\hat{W}_1/C_{\hat{W}_1}(\bar{A})) \geq 2n$ or $3n$. If (3.7) failed, we would thus have $m(\bar{A}) = 2n$ or $3n$, respectively, as $A \in \mathcal{A}(T)$, and, interchanging $\hat{W}$ and $\hat{W}^{\bar{a}}$ if necessary, $\dim(U_1/C_{U_1}(\bar{A})) = 1$, and $\dim(U_1'/C_{U_1'}(\bar{A})) = 1$ or 2, respectively. But since U_1 is a natural module, the first equation and the size of $\bar{A}$ force $\bar{L} \cong SL_3(2^n)$ and $\bar{A}$ to be the full stabilizer of the chain $V_1 > C_{V_1}(\bar{A}) > 1$. But then $\dim C_{V_1'}(\bar{A}) = 1$, contradiction. This proves (3.7) and completes the proof of the lemma.

Lemma 3.19: $\bar{L} \not\cong Sp_4(2)'$.

Proof: Otherwise we have $\bar{X} = \bar{L}_0\langle\bar{t}\rangle$, where $|\bar{X} : \bar{L}_0| = 2$ and $\bar{L}_0 \cong A_6$ or Σ_6. Moreover, by Lemma 3.15, $\bar{J} \leq \bar{L}_0$. If $\bar{L}_0 \cong \Sigma_6 (\cong Sp_4(2))$, we may repeat the argument of Lemma 3.18 to reach a contradiction. Thus $\bar{L}_0 = \bar{L} \cong A_6 \cong L_2(9)$.

Again let $A \in \mathcal{A}(T)$ with $\bar{A} \neq 1$, and choose $\bar{a} \in \bar{A}^\#$. Then $m(V/C_V(\bar{a})) \leq m(\bar{A}) \leq 2$, and there are conjugates $\bar{a}_1, \bar{a}_2$ of $\bar{a}$ such that $\bar{L} = \langle \bar{a}, \bar{a}_1, \bar{a}_2 \rangle$. Hence $m(V/C_V(\bar{L})) \leq 6$. On the other hand, $\bar{X}$ contains a Frobenius group of order $9 \cdot 8$ and so $m(V) \geq 8$ as V is a faithful $\bar{X}$-module. This contradiction proves the lemma.

We now complete the proof of the proposition. By the preceding two lemmas, $\bar{L} \cong 3A_6$. Choose $A \in \mathcal{A}(T)$ with $\bar{A} \neq 1$ and $\bar{A}$ minimal. The argument of Lemma 3.6(ii) may be repeated to give $[\bar{A}, Z(\bar{L})] = 1$. On the other hand, by Lemma 3.15, $\bar{A}\bar{L}/Z(\bar{L}) \cong A_6$ or Σ_6, and in the latter case just as in Lemma 3.9, $[\bar{A}, Z(\bar{L})] \neq 1$, contradiction. Therefore $\bar{A} \leq \bar{L}$. Since every involution of $\bar{L}$ inverts an element of order 5, $|\bar{A}| > 2$ by Lemma 3.2(ii). Hence $\bar{A} \cong E_4$. Thus $\bar{A}$ and $\bar{A}^{\bar{t}}$ are the two E_4-subgroups of $\bar{T} \cap \bar{L}$, and so are not $\bar{L}$-conjugate. But then there exists $\bar{g} \in \bar{L}$ such that $\bar{L} = \langle \bar{A}, \overline{A^{tg}} \rangle$. (Indeed for any $\bar{g} \in \bar{L}$, $\langle \bar{A}, \overline{A^{tg}} \rangle$ must contain a Sylow 2-subgroup of $\bar{L}$, so if $\langle \bar{A}, \overline{A^{tg}} \rangle < \bar{L}$, then $\langle \bar{A}, \overline{A^{tg}} \rangle \cong D_8$ or Σ_4, and as $\bar{A}$ and $\overline{A^{tg}}$ are not conjugate, one normalizes the other. Hence we need only choose $\bar{g}$ so that $\overline{A^{tg}} \not\leq N_{\bar{L}}(\bar{A})$.) Finally, as $|V/C_V(\bar{A})| \leq |\bar{A}| = 4$, we have $|V/C_V(\bar{L})| \leq 16$. But as in Lemma 3.9, $|[V, Z(\bar{L})]| \geq 2^6$, contradiction. This completes the proof of the proposition.

4. Components invariant under the Baumann subgroup

Again X is a K-group with $F^*(X) = O_2(X)$ having a singular 2-reduced setup, with accompanying notation as previously specified. In particular, $S = C_T(\Omega_1(Z(J)))$ is the Baumann subgroup of T. Our main result here is the following:

Proposition 4.1: *Let $\bar{L}$ be a component of $\bar{X}$ not centralized by $\bar{J}$. If $\bar{L} \cong L_2(2^n)$, $Sz(2^n)$, $(S)U_3(2^n)$, $A_{2n+1}(n \geq 2)$, or $3A_7$, then $\bar{S}$ normalizes $\bar{L}$ and $\bar{L} \cong L_2(2^n)$ or A_{2n+1}.*

The last assertion is immediate by Proposition 3.5, so we need only prove that $\bar{S}$ normalizes $\bar{L}$. We suppose false and choose a counterexample with $\bar{X}$ of minimal order.

We first prove

Lemma 4.2: $\bar{X} = \langle \bar{L}^{\bar{T}} \rangle \bar{T}$.

Proof: Otherwise $\bar{X} \neq \langle \bar{L}, \bar{T} \rangle$. We then apply Lemma 3.3 to $\bar{W} = \langle \bar{L}^{\bar{T}} \rangle$, with $\bar{B} = \bar{T}$ and conclude by the minimality of $\bar{X}$ that $\bar{W} \cong \tilde{W}_1$ and $\tilde{S}$ normalizes $\tilde{L}$, where $\bar{X}_1 = \langle \bar{L}, \bar{T} \rangle$, and $\tilde{X}_1$ is the factor group of $\bar{X}_1$ defined in Lemma 3.3. But then it is immediate that $\bar{S}$ normalizes $\bar{L}$, contrary to assumption. The lemma follows.

Next choose $A \in \mathcal{A}(T)$ as follows. If $\bar{J}$ does not normalize $\bar{L}$, choose $\bar{A} \not\leq N_{\bar{X}}(\bar{L})$; while if $\bar{J}$ normalizes $\bar{L}$, choose A so that $\bar{A} \neq 1$ and $\bar{A}$ is minimal subject to this condition. Replacing $\bar{L}$ by a $\bar{T}$-conjugate, we may assume that $[\bar{L},\bar{A}] \neq 1$. Again by the minimality of $\bar{X}$ and Lemma 3.3, we conclude

LEMMA 4.3: *Either $\bar{J}$ normalizes $\bar{L}$ or $\bar{X} = \langle \bar{L}^{\bar{A}}\rangle \bar{A}$..*

Set $\bar{A}_1 = N_{\bar{A}}(\bar{L})$. We next prove

LEMMA 4.4: *$\bar{A}_1$ does not centralize $\bar{L}$.*

Proof: Indeed, otherwise $\bar{A}_1 < \bar{A}$ since $[\bar{A},\bar{L}] \neq 1$, so $\bar{J}$ does not normalize $\bar{L}$. Hence by the preceding lemma, $\bar{X} = \bar{K}\bar{A}$, where $\bar{K} = \langle \bar{L}^{\bar{A}}\rangle$. Thus $\bar{A}_1 = C_{\bar{A}}(\bar{L}) = C_{\bar{A}}(\bar{K}) = 1$. By Lemma 3.2(ii3), $|\bar{A}| > 2$.

Let $\bar{a} \in \bar{A}^{\#}$ and set $\bar{I} = E(C_{\bar{K}}(\bar{a}))$. Then $\bar{A}/\langle\bar{a}\rangle$ permutes faithfully the components of $\bar{I}$. Applying Lemma 3.3 to $\bar{W} = \bar{I}$ and $\bar{B} = \bar{A}$, we conclude by the minimality of $\bar{X}$ that $\tilde{A}$ normalizes all components of $\tilde{W} \cong \bar{W}$ (where $\tilde{X}_1$ is as defined in Lemma 3.3). Therefore $\bar{A}$ normalizes all components of $\bar{W}$, so $\bar{A}/\langle\bar{a}\rangle = 1$, contradicting $|\bar{A}| > 2$, and the lemma is proved.

We next prove

LEMMA 4.5: *$\bar{J}$ normalizes $\bar{L}$.*

Proof: Suppose false, in which case again $\bar{X} = \bar{K}\bar{A}$, where $\bar{K} = \langle \bar{L}^{\bar{A}}\rangle$. By our choice of $\bar{A}$, $\bar{K} > \bar{L}$, so if we write $\bar{A} = \bar{A}_1 \times \bar{A}_2$, then $\bar{A}_2 \neq 1$ and $\bar{A}_2$ transitively permutes the components of $\bar{K}$. Thus $\bar{I} = E(C_{\bar{K}}(\bar{A}_2))$ is a diagonal of $\bar{K}$, so $\bar{I} \cong \bar{L}$ or $\bar{L}/Z(\bar{L})$. Furthermore, as $\bar{A}_1$ does not centralize $\bar{L}$, it does not centralize $\bar{I}$. We set $\bar{X}_1 = \bar{I}\bar{A}$, $\bar{R} = O_2(\bar{X}_1)$, $V_1 = C_V(\bar{R})$, and let X_1 be the preimage of $\bar{X}_1$ in X, $C_1 = C_{X_1}(V_1)$, and $\tilde{X}_1 = X_1/C_1$, so that V_1 is 2-reduced and singular in X_1 by Lemma 3.3 and $\tilde{I} = F^*(\tilde{X}_1) \cong \bar{I}$. By minimality of X, $\tilde{I} \cong \bar{L} \cong L_2(2^n)$ or A_{2n+1}.

Thus Proposition 3.12 applies to X_1 and V_1, so in either case V_1 has a unique nontrivial $\tilde{A}\tilde{I}$-composition factor U/U_0. In fact, we can take $U = [V_1,\tilde{I}]$ and $U_0 = C_U(\tilde{I})$. (By convention, when $\tilde{I} \cong L_2(4) \cong A_5$, we write $\tilde{I} \cong L_2(4)$ or A_5 to indicate the type of the module U/U_0.) Moreover, we have by Proposition 3.12:

$$U(A \cap C_1) \in \mathcal{A}(T). \tag{4.1}$$

We set $A^* = U(A \cap C_1)$.

Returning now to $\bar{X}$, we have $\overline{A^*} = \bar{A} \cap \bar{C}_1$ as $\bar{U} \leq \bar{V} = 1$. By our minimal choice of $\bar{A}$, either $\overline{A^*} = \bar{A}$ or $\overline{A^*} = 1$. However, as $\tilde{I} \cong \bar{I}$ and $\tilde{A}^* \leq \tilde{C}_1 = 1$,

$[\overline{A^*}, \tilde{I}] = 1$; that is, $\overline{A^*} \leq C_{\bar{A}}(\tilde{I}) = \bar{A}_2 C_{\bar{A}}(\bar{K})$. But $\bar{A}_1$ does not centralize $\bar{K}$ as $\bar{K} = \langle \bar{L}^{\bar{A}} \rangle$ and $\bar{A}_1$ does not centralize $\bar{L}$ by Lemma 4.4, so $\overline{A^*} < \bar{A}$. This forces $\overline{A^*} = 1$. But $\bar{A}_2 \leq \overline{A^*}$ by definition of $\tilde{I}$, so $\bar{A}_2 = 1$. Therefore $\bar{A}$ normalizes $\bar{L}$, contrary to the fact that $\bar{K} > \bar{L}$, and the proof is complete.

Thus $\bar{A}_1 = \bar{A}$ and $[\bar{L}, \bar{A}] = \bar{L}$ as $\bar{A}_1$ does not centralize $\bar{L}$. By Proposition 3.5, $\bar{L} \cong L_2(2^n)$ or A_{2n+1}. Applying Lemma 3.3 and Proposition 3.12 now as in the preceding lemma, with $\bar{L}\bar{A}$ as $\tilde{X}_1$, and $\bar{L}$ in the role of $\tilde{I}$, and defining $U = [\bar{L}, V_1]$, again (4.1) holds. Set $A^* = U(A \cap C_1) \in \mathcal{A}(T)$. The same argument as there shows that $\overline{A^*} = 1$. Thus V centralizes A^*, and it follows that $V \leq A^*$. But L and A normalize U, whence $[A, V] \leq [A, U(A \cap C_1)] \leq U$. Since $\bar{L} = [\bar{L}, \bar{A}]$, our argument therefore yields:

LEMMA 4.6: *$\bar{L}$ centralizes V/U, where $U = [V_1, \bar{L}] = [V, \bar{L}]$.*

Again we put $U_0 = C_U(\bar{L})$, so that U/U_0 is the unique nontrivial $\bar{L}$-composition factor within V. Also as $\bar{J}$ normalizes $\bar{L}$, $\bar{J}$ normalizes both U and U_0.

We next prove

LEMMA 4.7: *$U \cap E \not\leq U_0$.*

Proof: Recall that $E = \Omega_1(Z(J))$. If $\bar{L} \cong A_{2n+1}$, then $U_0 = 1$ by Proposition 3.12. But $C_U(J) \neq 1$ as J normalizes U, and as $U \leq A^* \leq J$, $1 \neq C_U(J) \leq E$, so the lemma holds in this case. Hence we can assume that $\bar{L} \cong L_2(2^n)$.

Set $T_0 = T \cap L$ and $Z_0 = C_U(T_0)$. Then by Proposition 3.12, $\tilde{A} = \tilde{T}_0$ (recall that $\tilde{L} = F^*(\tilde{X}_1)$) and $C_U(\tilde{T}_0) \not\leq U_0$. Trivially $C_U(\tilde{A}) = C_U(A)$ and $C_U(\tilde{T}_0) = C_U(T_0) = Z_0$, so we conclude that $C_U(A) = Z_0 \not\leq U_0$. Furthermore, the same conclusion holds if we replace A by any $B \in \mathcal{A}(T)$ for which $[\bar{B}, \bar{L}] \neq 1$. (Notice that although the definition of U, made with reference to $\tilde{X}_1$, depends *a priori* on A, $U = [V, \bar{L}]$ is independent of A.)

On the other hand, for any $B \in \mathcal{A}(T)$ for which $[\bar{B}, \bar{L}] = 1$, we have $[\bar{B}, U] \leq U_0$ by the irreducibility of $\bar{L}$ on U/U_0, so $[\bar{B}, U, \bar{L}] = 1$, whence $[U, \bar{L}, \bar{B}] = 1$ by the three subgroups lemma. But $U = [V, \bar{L}]$, so $U = [U, \bar{L}]$. Thus $[\bar{B}, U] = 1$. It follows that $[B, Z_0] = 1$. Hence Z_0 centralizes $J = \langle A \mid A \in \mathcal{A}(T) \rangle$, so $Z_0 \leq E$, and the lemma holds in this case as well.

Now we can quickly complete the proof. Since we are arguing by contradiction, there is $\bar{x} \in \bar{S}$ such that $\bar{L} \neq \bar{L}^{\bar{x}}$. Then $[\bar{L}, \bar{L}^{\bar{x}}] = 1$, so $\bar{L}^{\bar{x}}$ normalizes

$U = [V, \bar{L}]$ and $U_0 = C_U(\bar{L})$, and $[\bar{L}^{\bar{x}}, U] \leq U_0$. Using the three subgroups lemma as in the preceding lemma, we conclude that $[\bar{L}^{\bar{x}}, U] = 1$. Likewise $[\bar{L}, U^{\bar{x}}] = 1$. Therefore $U \cap U^{\bar{x}} \leq C_U(\bar{L}) = U_0$. But $U \cap E \leq U \cap U^{\bar{x}}$, so by the preceding lemma $U \cap U^{\bar{x}} \not\leq U_0$. This is a contradiction, and the proposition is proved.

We also need the following variation of the proposition.

LEMMA 4.8: *If $\bar{L}$ is a component of $\bar{X}$, and $\bar{L} \cong L_3(2^n)$ or $Sp_4(2^n)'$, then either $\bar{J}$ normalizes $\bar{L}$ or $\bar{L}$ centralizes $Z \cap V$.*

Proof: Suppose false, and let X be a counterexample with $|\bar{X}|$ minimal. Choose $A \in \mathcal{A}(T)$ with $\bar{A} \not\leq N_{\bar{X}}(\bar{L})$. Set $\bar{K} = \langle \bar{L}^{\bar{A}} \rangle$ and $\bar{X}_0 = \bar{K}\bar{A}$, let X_0 be the preimage of $\bar{X}_0$ in X and let $T_0 = T \cap X_0$, $V_0 = C_V(O_2(\bar{X}_0))$, so that as usual V_0 is 2-reduced in X_0, and $F^*(\tilde{X}_0) = \tilde{K} = \langle \tilde{L}^{\tilde{A}} \rangle \cong \bar{K}$, where $\tilde{X}_0 = X_0/C_{X_0}(V_0)$. Since $\bar{X}_0 = \bar{K}\bar{A}$, clearly $T_0 \in \mathrm{Syl}_2(X_0)$, and setting $Z_0 = \Omega_1(Z(T_0))$, we have $Z \cap V \leq Z_0 \cap V_0$, so $[\tilde{L}, Z_0 \cap V_0] \neq 1$. Hence X_0 is also a counterexample, so $X = X_0$ by the minimality of X. Thus $\bar{X} = \bar{K}\bar{A}$.

Set $A_1 = N_A(\bar{L})$ and write $A = A_1 \times A_2$. For any $A_0 \leq A_2$, we can set $\bar{I}_0 = E(C_{\bar{K}}(\bar{A}_0))$ and $\bar{X}_1 = \bar{I}_0\bar{A}$. Then, as usual, we obtain a 2-reduced V_1 in the preimage X_1 of $\bar{X}_1$ in X, with $\tilde{X}_1 = X_1/C_{X_1}(V_1) = \tilde{I}_0\tilde{A}$ and $F^*(\tilde{X}_1) = \tilde{I}_0$. If $|A_2| > 2$, we can take A_0 to be a hyperplane of A_2. Then $\tilde{A}$ permutes nontrivially the components of $\tilde{I}_0$, each of which is isomorphic to $\bar{L}$, contradicting the minimality of X. Hence $|A_2| = 2$. Now we take $A_0 = A_2$ and put $\bar{I} = \bar{I}_0$, so that $\bar{I} \cong \bar{L}$.

By Proposition 3.13, no element of A_1 induces a graph or graph-field automorphism on $\tilde{I}$. Let $\bar{M}_1$ and $\bar{M}_2$ be the maximal parabolic subgroups of $\bar{L}$ containing $\bar{T} \cap \bar{L}$, and set $\bar{P}_i = O^{2'}(\bar{M}_i)$, $i = 1, 2$. It follows that $\bar{A}_1$ normalizes $\bar{P}_1$ and $\bar{P}_2$. Set $\bar{R}_i = \langle \bar{P}_i^{\bar{A}_2} \rangle$, $i = 1, 2$, so that $\bar{K} = \langle \bar{R}_1, \bar{R}_2 \rangle$. Since $\bar{L}$ does not centralize $Z \cap V$ and $\bar{L} \leq \bar{K}$, some $\bar{R}_i$ does not centralize $Z \cap V$, $i = 1$ or 2. For simplicity, set $\bar{R} = \bar{R}_i$ for such a value of i, and put $Y = C_V(O_2(\bar{R}))$, $\bar{H} = \langle \bar{R}, \bar{T} \rangle = \langle \bar{R}, \bar{A} \rangle$, and let H be the preimage of $\bar{H}$ in X. Now $\bar{R} = (\bar{T} \cap \bar{R})O^2(\bar{R})$ so $O^2(\bar{R})$ does not centralize $Z \cap V$ and in particular does not centralize Y. But $O^2(\bar{R}) = O^2(\bar{H})$ and $O^2(\bar{R})O_2(\bar{R})/O_2(\bar{R})$ is the unique chief factor of $\bar{R}$ which is not a 2-group. Hence $C_{\bar{R}}(Y) = O_2(\bar{R})$. It follows that Y is 2-reduced in H, and in $\tilde{H} = H/C_H(Y)$, $\tilde{A}_2$ interchanges the two $\tilde{A}$-conjugates of $\tilde{P}_i$ whose direct product is $\tilde{R}$.

If $n > 1$, then as $\tilde{P}_i \cong L_2(2^n)$, this contradicts Proposition 4.1. Hence $n = 1$, so $\bar{X} \cong L_3(2) \wr Z_2$ or $\bar{X}$ is isomorphic to a subgroup of $\Sigma_6 \wr Z_2$ containing

$A_6 \wr Z_2$. Accordingly, $m(\bar{A}) \le 3$ or $m(\bar{A}) \le 4$.

Let W be a chief factor of X in V on which $\bar{L}$ acts nontrivially. Set $\bar{A}_2 = \langle \bar{a} \rangle$. If $W_1 = C_W(\bar{L}) \ne 1$, then as X leaves invariant both $W_1 W_1^{\bar{a}} (\le W)$ and $W_1 \cap W_1^{\bar{a}}$, it follows that $W_1 \cap W_1^{\bar{a}} = 1$ and that $W = W_1 \oplus W_1^{\bar{a}}$. Also $W_1^{\bar{a}} = C_W(\bar{L}^{\bar{a}})$ is $\bar{L}$-invariant as $\bar{L}$ centralizes $\bar{L}^{\bar{a}}$, so $W_1^{\bar{a}}$ is a faithful $\bar{L}$-module. Now $m(\bar{A}) \ge m(V/C_V(\bar{A})) \ge m(W/C_W(\bar{A})) = m(W/C_{W_0}(\bar{A}_1))$, where $W_0 = C_W(\bar{a})$ is a diagonal of $W = W_1 \oplus W_1^{\bar{a}}$. However, clearly $m(W/W_0) = m(W_0) = m(W_1) \ge 3$ or 4; i.e., $m(W/W_0) \ge m(\bar{A})$, and so we must have $C_{W_0}(\bar{A}_1) = W_0$. Thus $\bar{A}_1 = 1$, against Lemma 3.2. We conclude that $W_1 = 1$.

Suppose first that $\bar{A} \cap \bar{K} \ne 1$ and let $\bar{b} \in \bar{A} \cap \bar{K}^{\#}$, so that $\bar{b} = \bar{b}_1 \bar{b}_1^{\bar{a}}$ with $\bar{b}_1 \in \bar{L}^{\#}$. Let $\bar{W} = W \otimes_{\mathbf{F}_2} \bar{\mathbf{F}}_2$ and let Y be a simple $\bar{K}$-submodule of $\bar{W}$. Then $\dim_{\bar{\mathbf{F}}_2}([Y, \bar{b}]) \le \dim_{\bar{\mathbf{F}}_2}([\bar{W}, \bar{b}]) = m([W, \bar{b}]) \le m(V/C_V(\bar{A})) \le 3$ or 4, respectively. On the other hand, since $W_1 = 1$, $C_Y(\bar{L}) = 1$, and similarly $C_Y(\bar{L}^{\bar{a}}) = 1$, so $Y = Y_1 \otimes_{\bar{\mathbf{F}}_2} Y_2$, where Y_1 and Y_2 are nontrivial modules for $\bar{L}$ and $\bar{L}^{\bar{a}}$, respectively (see [S, Lemma 68]). Clearly $\dim_{\bar{\mathbf{F}}_2}(Y_i) \ge 3$ or 4, respectively, and a direct calculation with Jordan blocks shows that $\dim_{\bar{\mathbf{F}}_2}([Y, \bar{b}]) \ge 4$ or 6, respectively, a contradiction. Therefore $\bar{A} \cap \bar{K} = 1$, whence (by Lemma 3.2) $|\bar{A}| = 4$ and $\bar{L} \cong A_6$.

In this remaining case one easily finds four conjugates $\bar{B}_1, \bar{B}_2, \bar{B}_3, \bar{B}_4$ of $\bar{A}$ generating $\bar{X}$. Hence $C_W(\langle \bar{B}_1, \bar{B}_2, \bar{B}_3, \bar{B}_4 \rangle) = 1$, so $m(W) \le 4m(W/C_W(\bar{A})) \le 4m(\bar{A}) = 8$. However, $m(W) \ge 4^2 = 16$ since $\dim(Y_i) \ge 4$. This contradiction completes the proof of the lemma.

5. Generation of simple K-groups

To carry out an inductive argument we need information about generation of certain simple K-groups by subgroups containing a Sylow 2-subgroup.

To state the result we need a definition. Let K be a simple K-group, T a 2-subgroup of $\mathrm{Aut}(K)$, and Y a T-invariant subgroup of $\mathrm{Aut}(K)$. Let $\mathcal{N}(Y,T)$ be the set of T-invariant subgroups N of Y such that

(1) $O(N) = 1$; and
(2) No component of N is isomorphic to $L_2(2^n)$ or A_{2^n+1}, $n \ge 2$.

Set $N(Y,T) = \langle N \mid N \in \mathcal{N}(Y,T) \rangle$. We shall prove

PROPOSITION 5.1: *Let K be a simple K-group and T a 2-subgroup of* $\mathrm{Aut}(K)$. *Then $K = N(K,T)$ in each of the following cases:*

(1) *$K \in Chev(2)$ and K is not isomorphic to one of the groups $L_2(2^n), Sz(2^n)$, $(S)U_3(2^n), L_3(2^n)$, or $Sp_4(2^n)'$;*
(2) *$K \cong L_3(2^n)$ or $Sp_4(2^n)'$ and the image of T in $\mathrm{Out}(K)$ acts trivially on the Dynkin diagram of K;*
(3) *$K \cong A_{2n}$, $n \geq 4$; or*
(4) *$K \in Spor - \{J_1, Ly\}$.*

Proof: Without loss, we may replace T by TT^*, where T^* is a T-invariant Sylow 2-subgroup of K (this only shrinks $N(K,T)$) and assume $T \cap K \in \mathrm{Syl}_2(K)$. In (1) and (2), we consider the set $\mathcal{P}$ of parabolic subgroups P of K such that $P < K$ and P is T-invariant. For any $P \in \mathcal{P}$, $F^*(N_K(P)) = O_2(P)$ (see [GL; 13-14]), and so $N_K(P) \leq N(K,T)$. But in (1), since K is not one of the listed exceptional groups, the set of nodes of the Dynkin diagram Δ of K is the union of proper subsets invariant under $\mathrm{Aut}(\Delta)$ and so K is generated by the corresponding parabolics in $\mathcal{P}$, as required.

In (2), the hypothesis implies that T normalizes all parabolics of K containing $T \cap K$ and so again $K = \langle P \mid P \in \mathcal{P} \rangle = N(K,T)$.

In (3) and (4), set $Y = \mathrm{Aut}(K)$, so that $|Y : K| \leq 2$. Without loss, we may take $T \in \mathrm{Syl}_2(Y)$. Now if $N \in \mathcal{N}(Y,T)$, then clearly $N \cap K \in \mathcal{N}(K,T)$, and as $|N : N \cap K| \leq 2$ and $N \cap T \in \mathrm{Syl}_2(N)$, also $N = (N \cap T)(N \cap K)$. Thus $N(Y,T) \leq TN(K,T)$. Clearly also $N(K,T) \leq N(Y,T)$ so $N(Y,T) = N(K,T)T$. Hence it suffices to show that $Y = N(Y,T)$, for then $K \leq N(K,T)T$, whence $K = N(K,T)$.

If $K = A_{2n}$, then $Y = \Sigma_{2n}$ as $n \geq 4$. Let 2^m be the highest power of 2 with $2^m < 2n$. There is a subgroup $N_1 \cong \Sigma_a \times \Sigma_b$ of Y which is T-invariant, where $a = 2^m$ and $a + b = 2n$; moreover, N_1T is maximal in Y. Since a and b are even, $O(N_1) = 1$ and $N_1T \in \mathcal{N}(Y,T)$. Hence either $N(Y,T) = Y$ or $N(Y,T) = N_1T$. Thus we can assume that the latter holds.

Furthermore, there is $N_2 \cong E_{2^n} \cdot \Sigma_n$, where N_2 is the centralizer in Y of an involution without fixed points on $2n$ letters. Thus $F^*(N_2) = O_2(N_2)$. As $|Y : N_2|$ is odd, we may choose N_2 so that $T \leq N_2$, whence $N_2 \in \mathcal{N}(Y,T)$. Clearly N_2 is transitive on $2n$ letters and the only nontrivial system of imprimitivity for N_2 consists of the n orbits of $O_2(N_2)$. On the other hand, N_1T either is intransitive (if $a \neq b$) or is transitive with the only system of imprimitivity consisting of two sets of size a (if $a = b$). Since $n > 2$ it follows that $N_2 \not\leq N_1T$, so $Y = \langle N_1T, N_2 \rangle = N(Y,T)$ in this case as well.

Finally, assume that $K \in Spor - \{J_1, Ly\}$. We find $M_1, M_2 \in \mathcal{N}(Y,T)$ such that $K = \langle M_1, M_2 \rangle$. Except in the cases $K \in \{He, F_2, F_1\}$, we can use [Co] to find M_1 and M_2 as nonconjugate T-invariant maximal subgroups of K. In the three remaining cases take $M_1 = C_K(z)$, where $\langle z \rangle = Z(T \cap K)$, and $M_2 = N_K(U)$ for suitable $E_4 \cong U \triangleleft T \cap K$ with $N_K(U)/C_K(U) \cong \Sigma_3$, and set $K_1 = \langle M_1, M_2 \rangle$. Since $F^*(M_1) = O_2(M_1)$ in each case, also $F^*(C_K(U)) = O_2(C_K(U))$ and so $F^*(M_2) = O_2(M_2)$. In case $K \cong He$, we take U to be the strong closure of z in $Z(J(T \cap K)) \cong E_{16}$. Thus U is T-invariant. By [Co], M_1 is maximal in K in this case, so $K = K_1$, as required. In the other two cases, by [Co], M_1 is a maximal 2-local subgroup of K, so $O_2(K_1) = 1$. Likewise $O(M_1) = 1$, so $O(C_K(y)) = 1$ for all $y \in U^\#$, whence $O(K_1) = 1$. Thus $F^*(K_1) = L_1 \times \cdots \times L_s$, where the L_i are nonabelian simple groups. Now $M_1 = C_{K_1}(z)$ and $F^*(M_1)$ is extraspecial; it follows that $z = z_1 \cdots z_s$ where $z_i \in L_i$ and $F^*(C_{L_i}(z_i)) \le F^*(M_1)$. Clearly $|F^*(C_{L_i}(z_i))| \ge 4$, so since $F^*(M_1)' \cong Z_2$, $F^*(M_1)$ normalizes each L_i. Hence $z_i \in Z(F^*(M_1))$ for all i, so $s = 1$. Finally, since $M_1 = C_{K_1}(z)$ has a sporadic composition factor, $L_1 \notin Chev(2')$ by [GL; 14-1]; $L_1 \notin Chev(2)$ since otherwise $M_1 \cap L_1$ would be a parabolic subgroup of L_1 by the Borel-Tits theorem; $L_1 \notin Alt$ by a simple calculation; so $L_1 \in Spor$. Then $L_1 \cong K$ by [Co] and the structure of $C_{K_1}(z)$, so $K_1 = K$. This completes the proof.

For the final steps of the proof of the theorem we also need two generational results concerning alternating groups (cf. [As2; §7]). First, for any subgroups H and J of a group, and integer $k > 0$, set $[H, kJ] = [H, (k-1)J, J]$ (and $[H, 0J] = H$). Also set $H^{(k)} = [H, (k-1)H]$ for $k \ge 1$. Let S_k be a Sylow 2-subgroup of Σ_{2^k}. By [K], S_k has nilpotence class 2^{k-1}.

Our results are stated for our group X, with notation as set up in the Introduction.

PROPOSITION 5.2: *Assume that $C = Q$, $\bar{X} \cong \Sigma_{2n+1}$, and X contains a near component K of type A_{2n+1}. If I is a subgroup of X such that $\bar{I} \cong \Sigma_{2n}$ with $\bar{I}$ invariant under $\bar{T}$, then either $X = \langle I, C(X;T) \rangle$ or $2n+1 = 2^m + 1$ for some m.*

Proof: Assume that $2n+1 \neq 2^m + 1$. In particular, $2n+1 \ge 7$. Set $R = O_2(K)$ and $U = [R, K]$, so that U is a natural A_{2n+1}-module. As argued in the proof of Proposition 3.12, $H^1(\bar{K}, U)$ is trivial.

Let $\bar{X} = \Sigma_{2n+1}$ act on $\Omega = \{1, \ldots, 2n+1\}$, with notation chosen so that $\bar{T}$ fixes 1, and let Ψ_0 be the largest orbit of $\bar{T}$, of size 2^m, say. Set $\Psi_1 = \Psi - \Psi_0$, so that $1 \in \Psi_1$. Since $2n \neq 2^m$, $|\Psi_1| \geq 3$. Let $\bar{X}_i$, $i = 0,1$, be the pointwise stabilizer of Ψ_{1-i} in $\bar{X}$, so that $\bar{X}_i \cong \Sigma_{\Psi_i}$. Now $\bar{I}$ is the stabilizer of 1 in $\bar{X}$, and since $|\Psi_1| \geq 3$, it follows that $\langle \bar{X}_1', \bar{I} \rangle = \bar{X}$. Since $Q \leq T \leq C(X;T)$, it suffices to prove that $\bar{X}_1' \leq \overline{C(X;T)}$ in order to conclude that $X = \langle I, C(X;T) \rangle$.

Let T_∞ be the smallest nontrivial term of the lower central series of T and set $D = C_X(T_\infty)$. Thus $T \leq D \leq C(X;T)$, and we shall show in fact that $\bar{X}_1' \leq \bar{D}$, by induction on $|X|$.

If $T_\infty \leq Z(X)$, then $D = X$ and the desired assertion is trivial, so we may assume that $T_\infty \not\leq Z(X)$. Then $\tilde{T}_\infty \neq 1$ in $\tilde{X} = X/Z(X)$, so $\tilde{T}_\infty$ is the smallest nontrivial term of the lower central series of $\tilde{T}$. If $Z(X) \neq 1$, then by induction in $\tilde{X}$ the stabilizer of the chain $T_\infty Z(X) > Z(X) > 1$ covers $\bar{X}_1'$, and since $\bar{X}_1' = O^2(\bar{X}_1')$, $C_X(T_\infty Z(X))$ covers $\bar{X}_1'$, as desired. Therefore we may assume that $Z(X) = 1$.

Now $H^1(\bar{K}, U)$ is trivial, so $R = Z(K) \times U$. Since $X = KT$ and $Z(X) = 1$, it follows that $Z(K) = 1$, so $R = U$. Now every element of Q stabilizes the chain $K > U > 1$; and since $H^1(\bar{K}, U)$ is trivial, it follows that $Q = UC_Q(K)$. Again $C_Q(K) \triangleleft X = KT$, so as $Z(X) = 1$, $C_Q(K) = 1$. Thus $Q = U$. Set $U_1 = [\bar{X}_1, U]$, so that U_1 is a natural $\bar{X}_1$-module. Also set $U_0 = C_U(\bar{X}_1)$, so that $U = U_0 \times U_1$ and U_0 is the full permutation module for $\bar{X}_0 \cong \Sigma_{2^m}$. Since U_1 is absolutely irreducible for $\bar{X}_1$ and $[\bar{X}_0, \bar{X}_1] = 1$, we have $[\bar{X}_0, U_1] = 1$.

Let X_i be the preimage of $\bar{X}_i$ in X and set $T_i = T \cap X_i$, so that $T_i \in \mathrm{Syl}_2(X_i)$, $i = 0,1$. Thus $\bar{T} = \bar{T}_0 \times \bar{T}_1$. Moreover, U_0 is normalized by $\bar{X}_1$ and $\bar{X}_0$ by its definition, so $U_0 \triangleleft T$. Set $H = X_1X_0$ and $\hat{H} = H/U_0$. Now $\bar{T}_0 \cong S_m$ and $\bar{T}_1 = \bar{T}_{11} \times \cdots \times \bar{T}_{1s}$, where each $\bar{T}_{1i} \cong S_{k_i}$, and $|\Psi_1| = 2^{k_1} + \cdots + 2^{k_s}$, $m > k_1 > \cdots > k_s$.

Let V_1 be the $\mathbf{F}_2$-permutation module for $\bar{T}_1$. Then $\hat{U} = \hat{U}_1$ is isomorphic as $\bar{T}_1$-module to a submodule of V_1. Moreover, $V_1 = V_{11} \oplus \cdots \oplus V_{1s}$, where V_{1i} is the $\mathbf{F}_2$-permutation module for $\bar{T}_{1i}$ (on its nontrivial orbit), and $[V_{1i}, \bar{T}_{1j}]$ is trivial for all $i \neq j$. Since $m(V_{1i}) = 2^{k_i} \leq 2^{m-1}$ we have $[\hat{U}, 2^{m-1}\bar{T}_1] = 1$. By our remarks above, $\bar{T}_{1i}$ has class 2^{k_i-1}, so $\bar{T}_1$ has class at most 2^{m-2}. Therefore $\hat{T}_1^{(2^{m-2})} \leq \hat{U}$ and then $\hat{T}_1^{(2^{m-1}+2^{m-2}+1)} = 1$. Hence $\hat{T}_1^{(2^m-1)} = 1$ if $m \geq 3$. If $m = 2$, the same conclusion holds, since then $|\Psi_1| = 3$, so $|\bar{T}_1| = 2$, $|\hat{U}| = 4$, and thus $|\hat{T}_1| = 8$.

We also know that $\bar{T}_0 \cong S_m$ has class 2^{m-1}, so $\hat{T}_0^{(2^{m-1}+1)} \leq \hat{U}$; but $\hat{U} = \hat{U}_1$ and $[\bar{X}_0, U_1] = 1$, so $\hat{T}_0^{(2^{m-1}+2)} = 1$. In particular, $\hat{T}_0^{(2^m)} = 1$. Furthermore, since

$\bar{T} = \bar{T}_0 \times \bar{T}_1$, we have $[\hat{T}_0, \hat{T}_1] \le \hat{U}$, so $[\hat{T}_0, \hat{T}_1, \hat{T}_0] = 1$, and then $[\hat{T}_0, \hat{T}_0, \hat{T}_1] = 1$ by the three subgroups lemma.

We now have $[\hat{T}_0, \hat{T}_1, \hat{T}_0] = [\hat{T}_0, \hat{T}_0, \hat{T}_1] = \hat{T}_0^{(2^m)} = \hat{T}_1^{(2^m-1)} = 1$, and as $\hat{T} = \hat{T}_0\hat{T}_1$ with $\hat{T}_i \lhd \hat{T}$, $i = 0, 1$, it follows that $\hat{T}^{(2^m)} = 1$. Thus, $T^{(2^m)} \le U_0$. On the other hand, $\bar{T}_0 \cong S_m$ contains a 2^m-cycle, which acts freely on U_0, so $[U_0, (2^m - 1)T_0] \ne 1$. Therefore $T^{(2^m)} \ne 1$, so $T_\infty \le U_0$. This implies that $\bar{X}_1 \le \bar{D}$, and the proof is complete.

Finally we treat the exceptional A_7 case.

LEMMA 5.3: *Assume that $C = Q$ and $\bar{X} \cong A_7$. If X contains a near component K of small A_7 type, then $X = \langle C_X(Z), N_X(J)\rangle$.*

Proof: We may assume that $V = \langle Z^X \rangle$ is singular, otherwise the desired conclusion holds trivially. By Proposition 3.12, $\bar{J}$ is a root four-subgroup of $\bar{X}$, and so if we set $\bar{N} = N_{\bar{X}}(\bar{J})$, then $\bar{N}$ is a maximal subgroup of $\bar{X}$ of order $2^3 \cdot 3^2$. Clearly also $\bar{N} = \overline{N_X(J)}$. The desired conclusion is now equivalent to the statement that there exists $x \in C_X(Z)$ of odd order such that $\bar{x} \notin \bar{N}$.

As seen in Proposition 3.12, $H^1(\bar{X}, U)$ is trivial, where $U = [K, V]$. It follows that $V = U \times (V \cap Z(X))$. Passing to $\tilde{X} = X/V \cap Z(X)$, we may assume that $O_2(K) = U$ and must prove that $C_X(Z)$ contains an element x of odd order with $\bar{x} \notin \bar{N}$. But any $g \in Q$ stabilizes the chain $K > U > 1$, and since $H^1(\bar{X}, U)$ is trivial, it follows that $g \in KC_X(K)$. Thus $Q = U \times C_Q(K)$, and hence $Z = Z_0 \times C_Z(K)$, where $Z_0 = Z \cap U$. Therefore it is enough to show that some element x of odd order with $\bar{x} \notin \bar{N}$ centralizes Z_0.

However, $|Z_0| = 2$ (which can be seen by restriction to a Frobenius subgroup of $\bar{X}$ of order 20). By Proposition 3.12, $\bar{X}$ is transitive on $U^\#$, so $|\bar{X} : C_{\bar{X}}(Z_0)| = 15$. Hence we can take x of order 7, and the proof is complete.

6. Baumann's $L_2(2^n)$ lemma

In this section we establish a reduction lemma due to Baumann for X in the case that X has a singular 2-reduced setup and $\bar{X}$ has an $L_2(2^n)$ component. In the next section we establish an analogue for A_{2n+1} components due to Aschbacher [As2; 3.3].

We set up the situation for both results. Our hypothesis is the following:

(6.1) (1) $F^*(X) = O_2(X)$ and X has a singular 2-reduced setup;

(2) $Q \in \underset{2}{\mathrm{Syl}}(C)$;

(3) $A \in \mathcal{A}(T), \bar{A} \neq 1,$ and for all $B \in \mathcal{A}(T)$ with $1 \neq \bar{B} \leq \bar{A}$, we have $\bar{B} = \bar{A}$;

(4) $\bar{K}$ is a subgroup of $\bar{X}$ such that $[\bar{A}, \bar{K}] \neq 1,$ and either $\bar{K} \cong L_2(2^n)$ $(n \geq 1)$ or $\bar{K} \cong A_{2n+1}$ $(n \geq 2)$;

(5) Either $\bar{K}$ is a component of $\bar{X}$, or else $\bar{K} \cong L_2(2)$ and $\bar{K}$ is $\bar{S}$-invariant.

In this situation, we first observe that by (6.1)(5) and Proposition 4.1,

$$\bar{S} \text{ normalizes } \bar{K}. \tag{6.2}$$

We set

$$\bar{X}_0 = \overline{KS}, \tag{6.3}$$

and let X_0 and K be the preimages of $\bar{X}_0$ and $\bar{K}$ in X. Set

$$V_0 = C_V(O_2(\bar{X}_0)),\ C_0 = C_{X_0}(V_0), \text{ and } \tilde{X}_0 = X_0/C_0, \tag{6.4}$$

so that by Lemma 3.3, $\tilde{K} \cong \bar{K}$ and V_0 is 2-reduced and singular in X_0 (with $[\tilde{A}, \tilde{K}] \neq 1$).

Also put

$$U = [V_0, \tilde{K}] \text{ and } U_0 = C_U(\tilde{K}). \tag{6.5}$$

Then $\tilde{K}$, V_0, and U satisfy either (i), (ii), or (iii) of Proposition 3.12, or else $\tilde{K} \cong L_2(2)$, in which case Proposition 3.4 applies. We consider case (i) and the $L_2(2)$ case in this section, and the cases (ii) and (iii) in the next.

Proposition 6.1: *Assume (6.1), with either $\tilde{K}$ satisfying Proposition 3.12(i) or $\tilde{K} \cong L_2(2)$. Then X has a subgroup Y such that*

(i) *S normalizes Y and $S \cap Y \in \mathrm{Syl}_2(Y)$;*

(ii) *$\bar{Y} = \bar{K}'$, and $\bar{S}$ induces inner automorphisms on $\bar{K}$;*

(iii) *$F^*(Y) = O_2(Y)$ and $Y = O^2(Y)$; and*
(iv) *If $\bar{K}$ is subnormal in $\bar{X}$, then Y is subnormal in X.*

Proof: By Proposition 3.12(i), $\tilde{S}$ induces inner automorphisms on $\tilde{K}$ and

$$\tilde{A} = \tilde{J} = \tilde{S} \in \mathrm{Syl}_2(\tilde{K}). \tag{6.6}$$

(These assertions are trivial if $\tilde{K} \cong L_2(2)$.)

Since $X_0 = KS$, it follows that

$$\tilde{X}_0 = \tilde{K} \text{ and } \bar{X}_0 = \bar{K} \times O_2(\bar{X}_0). \tag{6.7}$$

Set $A_0 = (A \cap C_0)V_0$. By Propositions 3.12(i) and 3.4,

$$A_0 \in \mathcal{A}(T). \tag{6.8}$$

But clearly $\bar{A}_0 < \bar{A}$ since $\bar{A}_0$ centralizes $\bar{K}$. Thus by (6.1)(3), $\bar{A}_0 = 1$. Now by (6.1)(2),

$$A_0 \leq Q, \text{ so } A_0 = (A \cap Q)V_0. \tag{6.9}$$

Since $V \leq Z(Q)$, (6.8) and (6.9) imply

$$V \leq A_0, \text{ so } V = (A \cap V)V_0.$$

By Propositions 3.4 and 3.12(i),

$$\text{(6.10)} \qquad \begin{aligned} &(1)\ U/U_0 \text{ is a natural } \bar{K} \cong L_2(2^n)\text{-module; and} \\ &(2)\ C_U(\bar{J}) \not\leq U_0. \end{aligned}$$

On the other hand, by (6.6) and (6.7), for some $w \in K$, $\langle A, A^w \rangle$ covers $\tilde{K}$, and then $\bar{K}' \leq \langle \bar{A}, \bar{A}^{\bar{w}} \rangle$. Hence $C_V(\bar{K}) \geq C_V(\bar{A}) \cap C_V(\bar{A})^{\bar{w}}$. Since $\bar{A}_0 = 1$, $|\bar{A}| = 2^n$, so $|V : C_V(\bar{A})| \leq 2^n$. Hence $|V : C_V(\langle \bar{A}, \bar{A}^{\bar{w}} \rangle)| \leq |V : C_V(\bar{K}')| \leq 2^{2n} = |U/U_0|$. It follows that

$$V = UC_V(\bar{K}') = UC_V(\langle \bar{A}, \bar{A}^{\bar{w}} \rangle); \tag{6.11}$$

and, in particular,

$$U = [V, \bar{K}'].$$

Set $T_0 = T \cap X_0 \in \mathrm{Syl}_2(X_0)$ and $B = \langle A^{T_0} \rangle$, and let L be the subnormal closure of B in X_0; i.e., the largest subgroup of X_0 such that $L = \langle B^L \rangle$. Set $Y = O^2(L)$. We shall argue that Y satisfies the conclusions of the proposition.

Notice that T_0 normalizes B and hence L. Thus $Y = O^2(LT_0)$. Also, since $[\bar{A}, \bar{K}'] \neq 1$, $\bar{K}' \leq \langle \bar{A}^{\bar{K}'} \rangle$ and so $\bar{K}' \leq \bar{L}$.

Set $W = \langle E^L \rangle$ (recall that $E = \Omega_1(Z(J))$). Clearly W is T_0-invariant. We first prove

LEMMA 6.2: *The following conditions hold:*

(i) $U \leq W \leq Q$ *and* W *is elementary abelian;*

(ii) $W = UE$; *and*

(iii) $S \cap Q \triangleleft LT_0$.

Proof: By (6.8) and (6.9), $A_0 \in \mathcal{A}(T)$ and $A_0 \leq Q$. Hence $\Omega_1(C_T(A_0)) \leq A_0 \leq Q$, and so $E \leq A_0 \leq Q$. Similarly, $A_0 \in \mathcal{A}(T^y)$ for any $y \in L$ and so $E^y \leq A_0 \leq Q$. Thus $W = \langle E^L \rangle \leq A_0$ and W is elementary abelian.

Set $F = E \cap U$. By (6.10)(2), $F \not\leq U_0$. But $\bar{K}' \leq \bar{L}$, and so $\langle F^L \rangle \geq \langle F^{\bar{K}'} \rangle$, which covers U/U_0. However, $\langle F^L \rangle \leq \langle E^L \rangle = W$, so W covers U/U_0. Since $U = [V, \bar{K}']$ and $U \cap W$ is invariant under $\bar{K}'(\leq \bar{L})$, $U \leq W$, and (i) holds.

Next, by (6.9), $A_0 = (A \cap Q)V_0$, so by (6.11), A centralizes A_0/U. Since $W \leq A_0$, A centralizes W/U. Conjugating by T_0 yields that B centralizes W/U, and so also $L = \langle B^L \rangle$ does. Hence by definition of W, $W = EU$, proving (ii).

Finally, as $U \leq V \leq Z(Q)$, $C_Q(W) = C_Q(E)$ by (ii). But $S = C_T(E)$, so $S \cap Q = C_Q(E) = C_Q(W)$. Since $W \triangleleft LT_0$ and $Q \triangleleft X$, $S \cap Q \triangleleft LT_0$, so (iii) holds, and the lemma is proved.

We set $R = S \cap Q$, so that $R \triangleleft LT_0$. Obviously $[S, Q] \leq S \cap Q = R$, so S centralizes Q/R. But $LS = \langle S^{LS} \rangle$ as $B \leq J \leq S$ and $L = \langle B^L \rangle$, so LS centralizes Q/R. Hence if we set $L^*T_0^* = LT_0/R$, then

$$L^*S^* \text{ centralizes } Q^*.$$

We next prove

LEMMA 6.3: *The following conditions hold:*

(i) Y *is subnormal in* K;

(ii) $F^*(Y) = O_2(Y)$; *and*

(iii) $\bar{Y} = \bar{K}'$.

Proof: By construction, $Y \triangleleft L$ and L is subnormal in X_0. Moreover, as $\bar{X}_0 = \bar{K} \times O_2(\bar{X}_0)$ by (6.7), $\bar{Y} = O^2(\bar{X}_0) = O^2(\bar{K}) = \bar{K}'$, so (iii) holds, and (i) follows immediately.

If $1 \neq x \in L$ has odd order, x does not centralize R, as x centralizes $Q^* = Q/R$, but not $F^*(X) = Q$. But $R \leq S$ and $R \triangleleft LT_0$, so $[R,x] \leq O_2(Y)$. Since $[R,x,x] = [R,x]$, x does not centralize $O_2(Y)$. Thus $C_Y(O_2(Y))$ is a 2-group, whence $C_Y(O_2(Y)) \leq O_2(Y)$ and $F^*(Y) = O_2(Y)$, proving (ii).

In view of the preceding lemma, it remains only to prove that $Y \cap S \in \mathrm{Syl}_2(Y)$ to complete the proof of the proposition.

We treat the cases $n > 1$ and $n = 1$ separately, first proving

LEMMA 6.4: *If $n > 1$, then $S \cap Y \in \mathrm{Syl}_2(Y)$.*

Proof: We argue first that $N_Y(T_0)$ contains a subgroup H of odd order such that $\bar{H} \cong Z_{2^n-1}$ is a Cartan subgroup of $\bar{K}$. Indeed, Y is T_0-invariant and $\bar{Y}\bar{T}_0 = \bar{K}\bar{T}_0 = \bar{X}_0$, so by the Frattini argument YT_0 contains such a subgroup H, and then $H \leq O^2(YT_0) = Y$.

Now $[S,H] \leq [T_0,Y] \leq Y$. But since $S = C_{T_0}(E)$ and E char J char T_0, H normalizes S, so $[S,H] \leq Y \cap S$. On the other hand, since $\bar{A} \leq \bar{S}$ and $[\bar{A},\bar{K}] = \bar{K}$, $[\bar{H},\bar{S}] \in \mathrm{Syl}_2(\bar{K})$. But $\bar{K} = \bar{Y}$ (as $n > 1$) and so $\bar{Y} \cap \bar{S} \in \mathrm{Syl}_2(\bar{Y})$. Thus to complete the proof, it suffices to show that if we set $C_1 = C \cap Y$, then $R \cap C_1 \in \mathrm{Syl}_2(C_1)$.

Set $Q_1 = Q \cap C_1$. Since Y^* centralizes $Q_1^* \leq Q^*$, so does C_1^*. Hence $Q_1^* \leq Z(C_1^*)$, so Q_1^* is abelian. Since $Q_1^* \in \mathrm{Syl}_2(C_1^*)$, Burnside's theorem implies that C_1^* has a normal 2-complement D^*. Set $\tilde{Y}^* = Y^*/D^*$, so that $\tilde{Q}_1^* \leq Z(\tilde{Y}^*)$ and $\tilde{Y}^*/\tilde{Q}_1^* \cong \bar{K} \cong L_2(2^n)$. But $\tilde{Y}^* = O^2(\tilde{Y}^*)$, so either $\tilde{Q}_1^* = 1$ or $n = 2$ and $\tilde{Y}^* \cong 2L_2(4) (\cong 2A_5)$. However, if we set $S_0 = S \cap Y$, we have seen that $\bar{S}_0 \in \mathrm{Syl}_2(\bar{Y})$, while $(S_0 \cap C)^* \leq R^* = 1$. Therefore $\tilde{S}_0^* \cong \bar{S}_0 \cong E_{2^n}$. But $2L_2(4)$ has no E_4-subgroup, so $\tilde{Y}^* \ncong 2L_2(4)$. We conclude that $Q_1^* = 1$, whence $R \in \mathrm{Syl}_2(C_1R)$, so $R \cap C_1 \in \mathrm{Syl}_2(C_1)$, and the lemma is proved.

The remaining case $n = 1$ is easier. With notation as above, as $\bar{L} \geq \bar{K}'$ and $\bar{K} \cong \Sigma_3$, $Z_3 \cong \bar{Y} = [\bar{L},\bar{A}]$. It follows that $\tilde{Y}^* \cong Z_3$ with Y^* centralizing Q_1^*, so $Y^* = O^2(Y^*)$ is of odd order and hence $Q_1^* = 1$. It follows that $R \cap Y = S \cap Y \in \mathrm{Syl}_2(Y)$, as required. With Lemmas 6.3 and 6.4, we conclude that Proposition 6.1 holds in all cases.

7. Aschbacher's A_{2n+1} analogue

We prove here an analogue of Proposition 6.1 for alternating components due to Aschbacher [As2; 3.3]. We continue the notation of the previous section.

PROPOSITION 7.1: *Assume (6.1), and suppose that $\tilde{K}$, V_0 and U_0 satisfy either (ii) or (iii) of Proposition 3.12. (See (6.2) — (6.5).) Then the following conditions hold:*

(i) *X has an S-invariant subnormal subgroup Y such that*
  (1) *$Y \cap S \in \mathrm{Syl}_2(Y)$;*
  (2) *$\bar{Y}\bar{S}/O_2(\bar{Y}\bar{S}) \cong \Sigma_3$ (case (ii)) or A_5 (case (iii));*
  (3) *$F^*(Y) = O_2(Y)$; and*
  (4) *$\bar{K} \le \langle \bar{I}, \bar{Y} \rangle$, where $\bar{I}$ is a $(\bar{T} \cap \bar{K})$-invariant subgroup of $\bar{K}$ isomorphic to A_{2n}; and*

(ii) *If $Y = HC(Y;S)$ for some normal near component H of Y such that $\bar{H} = \bar{Y}$ with H of type $L_2(2)$ or $L_2(4)$, respectively, then*
  (1) *$K = LC$, where L is a near component of X of type A_{2n+1} (case (ii)) or of small A_7 type (case (iii)); and*
  (2) *T has a characteristic subgroup P such that P normalizes L and $P \cap L \in \mathrm{Syl}_2(L)$.*

Proof: First suppose that case (ii) holds, so $\tilde{X}_0 \cong \Sigma_{2n+1} = \Sigma_\Omega$, where we may take $\Omega = \{0, 1, \cdots, 2n\}$. By Proposition 3.12, $V_0 \le J$, so $C_{V_0}(\tilde{J}) \le E = \Omega_1(Z(J))$. Thus by (7), (3), and (4) of Proposition 3.12(ii), $\tilde{S} = \tilde{J}$ is generated by n disjoint transpositions, say $(12), \cdots, (2n-1\ 2n)$, and for some $a \in A^{\#}$, $\tilde{a} = (12)$ (say), and a acts on V_0 as a transvection. Furthermore, as in (6) of that proposition, $B = \langle a, A \cap C_0, C_V(a) \rangle \in \mathcal{A}(T)$. Clearly $\bar{a} \in \bar{B} \le \bar{A}$, so by (6.1)(3), $\bar{B} = \bar{A}$. Hence $\bar{A} = \langle \bar{a} \rangle \times C_{\bar{A}}(\bar{K})$. Likewise $\langle C_A(\bar{K}), V \rangle \in \mathcal{A}(T)$ and (6.1)(3) thus implies that $\bar{A} = \langle \bar{a} \rangle$.

Set $\tilde{D} = \langle (012) \rangle$. Then clearly $\tilde{S}$ normalizes $\tilde{D}$, $\tilde{S}\tilde{D}/O_2(\tilde{S}\tilde{D}) \cong \Sigma_3$, and $\tilde{K} = \langle \tilde{D}, \tilde{I} \rangle$, where $\tilde{I} \cong A_{2n}$ is the stabilizer in $\tilde{K}$ of 0. Let $\bar{D}$ be the Sylow 3-subgroup of the preimage in $\bar{X}_0$ of $\tilde{D}$. Thus $|\bar{D}| = 3$, $\bar{S}\bar{D}/O_2(\bar{S}\bar{D}) \cong \Sigma_3$, and $\bar{K} = \langle \bar{D}, \bar{I} \rangle$, where $\bar{I} \le A_{2n}$, $\bar{I} \le \bar{K}$, and $\bar{I}$ maps on $\tilde{I}$.

Now $\langle \bar{a} \rangle \bar{D}$ is an $\bar{S}$-invariant subgroup of $\bar{X}$ isomorphic to $L_2(2)$, and $\langle \bar{a} \rangle = \bar{A}$ with $A \in \mathcal{A}(T)$. Thus Proposition 6.1 applies and yields the existence of a group Y such that Y is S-invariant, $\bar{Y} = \bar{D}$, $Y \cap S \in \mathrm{Syl}_2(Y)$, and $F^*(Y) = O_2(Y)$. Therefore (i) holds in this case.

In case (iii) we argue similarly. This time $\tilde{A} = \tilde{J} = \tilde{S}$ is a root four-group in $\tilde{K} \cong A_7$ by Proposition 3.12, and we similarly find $\bar{A} \cong E_4$.

Let $\tilde{D}$ and $\tilde{I}$ be root A_5 and A_6 subgroups of $\tilde{K}$ respectively (i.e., two-point and one-point stabilizers in $\tilde{K}$ on Ω), containing $\tilde{S}$ and invariant under $\tilde{T} \cap \tilde{K}$. Then $\tilde{K} = \langle \tilde{D}, \tilde{I} \rangle$. Let $\bar{D}$ be the layer of the preimage of $\tilde{D}$ in $\bar{X}_0$, and define $\bar{I}$ similarly, so that $\bar{D} \cong A_5$, $\bar{I} \cong A_6$, $\bar{D} \leq \bar{K}$, and $\bar{K} = \langle \bar{D}, \bar{I} \rangle$.

Since $[\bar{A}, \bar{D}] = \bar{D}$, Proposition 6.1 applies as in case (ii) and yields (i) in this case as well.

Suppose now that $Y = HC(Y;S)$ and $\bar{H} = \bar{Y}$, as in (ii). For any $x \in S$, H^x maps on $\bar{H} = \bar{Y}$, so $[H^x, V] = [H, V]$. Hence by Proposition 2.1, $H^x = H$; i.e., H is S-invariant. Also, we know that $(A \cap C)V \in \mathcal{A}(T)$, so $V \leq J \leq S$. Thus $[\bar{Y}, V] = [H, V] \leq [H, S] \leq H$, and as H is a near component, it follows that $[H, O_2(H)] = [\bar{Y}, V] \leq V$. Furthermore, $\bar{H} \leq \langle \bar{A}^{\bar{H}} \rangle$, so as H is a near component, $H \leq \langle A^H \rangle$.

Set $R = S \cap Q$ and $X^* = X/V$. Since S normalizes H, $[R, H] \leq O_2(H)$ and so $[R, H, H] \leq [O_2(H), H] \leq V$. Thus $[R^*, H^*, H^*] = 1$, so $[R^*, H^*] = 1$ as $H = O^2(H)$. Likewise $[J, Q] \leq R$ as $J \leq S$ and $J \triangleleft T$. Thus J^* and hence A^* centralizes Q^*/R^*, so $H^* \leq \langle A^{*H^*} \rangle$ does as well. We conclude that H^* centralizes Q^*. Since $\bar{K} \cong A_{2n+1}$ is simple and $\bar{H} \leq \bar{K}$, it follows now that $K_1^* = C_{K^*}(Q^*)$ covers $\bar{K}$. The preimage K_1 of K_1^* in X is normal in K. Also $C \cap K_1 \leq Q$ since any x of odd order in $C \cap K_1$ centralizes both Q^* and V and hence centralizes Q, so $x = 1$. We conclude that $K_1Q/Q \cong \bar{K}$.

Set $L = O^2(K_1)$. Then as K_1^* centralizes Q^*, L^* is a covering of $\bar{K}$, so $L^* \cong (2)A_{2n+1}$ (cf. [GL; 6-1]). Furthermore, $\bar{L} = \bar{K}$, so $K = LC$ and $[V, L] = [V, \bar{K}] = U$. By definition, L centralizes Q/V, so $L = O^2(L)$ centralizes Q/U. Also $L \triangleleft K$ and hence L is subnormal in X. Thus L is a near component of X and $K = LC$, so (ii1) holds.

Finally, set $F = E \cap U$ and $W = C_E(L)$. Since $V \leq J$, $E \leq C$, so $E \leq Q$ by (6.1)(2). Therefore $[L, EV] \leq [L, Q] = U$. But since $L = O^2(L)$ and $H^1(L/O_2(L), U)$ is trivial, as noted in the proof of Proposition 3.12, this implies that $EV \leq UC_{EV}(L) = U \times C_{EV}(L)$. Since J normalizes L, both factors are J-invariant. Also $E = \Omega_1(Z(J)) = C_{EV}(J)$ since $EV \leq J$. Therefore $E = (E \cap U) \times C_E(L) = F \times W$.

Now set $m = m(F)$ and define the subgroup P of T as follows:

$$P = \langle t \in T \mid m(E/C_E(t)) < m \rangle.$$

Since E char T, clearly P char T. We argue that (ii2) holds for P. Indeed, if $t \in T$ moves L, then by Proposition 2.1, L^t centralizes U, whence U^t is disjoint from U. In particular, $F \cap F^t = 1$, so $m(E/C_E(t)) \geq m = m(F)$. Thus $t \notin P$ and consequently P leaves L invariant. On the other hand, if $t \in T \cap L$, then t leaves F invariant, so $C_F(t) \neq 1$. Since $E = F \times W$ and t centralizes W, it follows that $m(E/C_E(t)) < m$, so $t \in P$. Thus $P \cap L = T \cap L \in \mathrm{Syl}_2(L)$, as required. This completes the proof of Proposition 7.1.

8. Normal subgroups having Aschbacher form

For the balance of the paper we let G be a minimal counterexample to Theorem A. Then G does not have Aschbacher form and, in particular, G has a singular 2-reduced setup with $V = \langle Z^G \rangle$. Furthermore, if H is any T-invariant subgroup of G with $Q \leq H$ and $HT < G$, then H does have Aschbacher form relative to T.

More generally, if H is an R-invariant subgroup of G for some $R \leq T$ such that $R \cap H \in \mathrm{Syl}_2(H)$, $F^*(H) = O_2(H)$, and $HR < G$, then H has Aschbacher form (relative to R) - i.e., $H = K(H)C(H;R)$. This observation will be used when R char T (in particular, with $R = S$), in which case $C(H;R) \leq C(G;T)$.

For brevity, for any T-invariant subgroup Y of G, we set $Y_0 = C(Y;T)$. In particular, $G_0 = C(G;T)$.

In this section we prove

PROPOSITION 8.1: *If $H \lhd G$ with $Q \leq H$ and $HT < G$, then $Q \in \mathrm{Syl}_2(H)$.*

As noted above, H has Aschbacher form relative to T, so that $H = K(H)H_0$. Since $H \lhd G$, obviously $K(H) \leq K(G)$.

Set $R = T \cap H$, so that $R \in \mathrm{Syl}_2(H)$ and $Q \leq R$. We shall establish the proposition by contradiction. Thus we assume

$$(8.1) \qquad Q < R.$$

Furthermore, we choose H so that R has maximal order.

Set $N = N_G(R)$, so that $G = HN$ by the Frattini argument as $H \lhd G$. Since $R \leq O_2(N)$ and $Q = O_2(G)$ with $Q < R$, we have $N < G$. Also $T \leq N$ as $R \lhd T$. Thus $NT < G$ and so likewise $N = K(N)N_0$. If $N = N_0$, then $G = HN_0 = K(H)H_0N_0 = K(H)G_0 = K(G)G_0$, contrary to assumption. Therefore $N \neq N_0$.

Let K_i, $1 \le i \le r$, be the near components of $K(N)$ not contained in N_0. Then $K = K_1K_2 \cdots K_r \triangleleft K(N)N_0 = N$, in view of Proposition 2.1, and $K_i \not\le N_0$ for all i.

Suppose some nontrivial product I of nonsolvable near components of K is normal in N with $I < K$. Then $HI \triangleleft HN = G$ and $HIT < G$. However, as I is nonsolvable, $T \cap HI > T \cap H = R$, so HI contradicts our maximal choice of H. It follows that no such product I exists. We conclude from this that the K_i are isomorphic, $1 \le i \le r$, and one of the following holds:

(8.2)
(1) N permutes the K_i transitively, and K_i is nonsolvable, $1 \le i \le r$; or
(2) $K_i \cong A_4, 1 \le i \le r$.

Moreover, in the latter case, Proposition 3.4 implies:

(8.3) If $K_i \cong A_4$, $1 \le i \le r$, then $K_i = [K_i, J]$.

We also immediately obtain

LEMMA 8.2: *Some K_i is not a near component of G.*

Proof: Indeed, $G = HN = HKN_0 = KHN_0$ as $H \triangleleft G$, whence

$$G = KK(H)H_0N_0 = KK(H)G_0.$$

If the lemma is false, then $KK(H) \le K(G)$, whence $G = K(G)G_0$, contradiction, so the lemma holds.

We establish the proposition by contradicting this conclusion. Notice that $\bar{K}_i \ne 1$; i.e., $[K_i, V] \ne 1$, for otherwise $K_i \le C_G(Z)$ and so $K_i \le N_0$, contradiction. But $V \le Q \le R \le O_2(N)$, and so $[K, O_2(N)] = [K, R] = [K, V]$. Hence if we put $G^* = G/V$, it follows that K^* centralizes R^*.

We first prove

LEMMA 8.3: *K^* centralizes $E(H^*)$.*

Proof: Let L^* be a component of H^*. Then $P^* = R^* \cap L^* \in \mathrm{Syl}_2(L^*)$. Since K^* centralizes R^* and induces a group of permutations of the components of H^*, K^* leaves L^* invariant.

If K^* is nonsolvable (whence K^* is perfect), K^* induces inner automorphisms on L^* by the Schreier property for simple K-groups (see [GL; 7-1]). But then as K^* centralizes $P^* \in \mathrm{Syl}_2(L^*)$, K^* centralizes L^*. Hence K^* centralizes L^* in this case.

Suppose then that K^* is solvable and assume by way of contradiction that, say, K_1^* does not centralize L^*. Then $[L^*, K_1^*] = L^*$. On the other hand, L^* does not centralize V; for if it did, then L^* would stabilize the chain $Q \geq V \geq 1$ (as $L^* \leq E(H^*)$), a contradiction as $L^* = O^2(L^*)$ and $Q = F^*(G)$. Therefore, if we let L be the preimage of L^* in G, then $\bar{L} \neq 1$.

By (8.3), $[K_1, A] = K_1$ for some $A \in \mathcal{A}(T)$. Hence $[\bar{K}_1, \bar{A}] = \bar{K}_1$. Since $1 \neq \bar{L} = [\bar{L}, \bar{K}_1]$, we conclude that $[\bar{L}, \bar{A}] \neq 1$. This implies that $C_{\mathrm{Aut}(L^*)}(P^*)$ is a 2-group. Indeed, if $\bar{L}$ is of Lie type, then by Proposition 3.5, $\bar{L} \in Chev(2)$ and the assertion holds by [GL, 13-3]. If $\bar{L} \in Spor$, it holds by the centralizer structures given in [GL, §5] (and induction) and for $\bar{L} \in Alt$, it follows for example from the fact that a Sylow 2-subgroup of A_n has orbits whose sizes are distinct powers of 2, so its centralizer in Σ_n acts semiregularly on each such orbit. Finally, K^* centralizes P^*, so $K^* = O^2(K^*)$ centralizes L^*, contradiction. The lemma follows.

We next prove

LEMMA 8.4: *K^* does not centralize $O(H^*)$.*

Proof: Suppose false. Since K^* centralizes $E(H^*)$ by the preceding lemma and also centralizes $O_2(H^*) \leq R^*$, K^* centralizes $F^*(H^*)$, whence $C_{H^*K^*}(F^*(H^*)) = K^*Y^*$, where $Y^* = Z(F^*(H^*))$. But $H^*K^* \triangleleft G^*$ as $H \triangleleft G$ and $K \triangleleft N$ with $G = HN$. Also $F^*(H^*) \triangleleft G^*$, so $K^*Y^* \triangleleft G^*$ with K^* centralizing Y^*.

If K^* is nonsolvable, then $K^* = E(K^*Y^*) \triangleleft G^*$, whence $KV \triangleleft G$. But K is a product of near components of KV as $V \leq R$, so K is a product of near components of G, contrary to Lemma 8.2. Thus K is solvable, whence K^* is an elementary abelian 3-group and K^*Y^* is abelian. Hence each K_i^* is subnormal in G^*, whence K_iV is subnormal in G. Again K_i is subnormal in K_iV, so each K_i is a near component of G, again contradicting Lemma 8.2. The lemma follows.

Set $D^* = O(H^*)$. Then D^* char $H^* \triangleleft G^*$, whence $[D^*, Q^*] \leq D^* \cap Q^* = 1$. Since $C_G(Q) \leq Q$, it follows that D^* acts faithfully on V. Likewise K^* centralizes $Q^* \leq R^*$, so $C_{K^*D^*}(V) \leq O_2(K^*)$.

We separate the solvable and nonsolvable cases, first eliminating the latter case.

LEMMA 8.5: *K is solvable.*

Proof: Suppose false. Since N^* permutes the K_i^* transitively, no K_i^* centralizes D^*, $1 \le i \le r$. Thus $C_{K^*}(D^*) \le Z(K^*)$. Furthermore, as no $K_i \le N_0$, J^* centralizes no K_i^* (otherwise the Frattini argument would yield that any such K_i lies in $N_G(J) \le N_0$). Thus $[K^*, J^*] = K^*$.

Set $T_1 = C_T(D^*)$. Since $D^* \triangleleft G^*$, we have $[T_1^*, K^*] \le C_{K^*}(D^*) \le Z(K^*)$. Hence $[T_1^*, K^*] = 1$, so $T_1 = C_T(D^* K^*)$.

Let D be the preimage of D^* in G. Since $C_{K^*D^*}(V) \le O_2(K^*)$, $\bar{D} = D^*$. By the previous paragraph, $\bar{T}_1 = C_{\bar{T}}(\bar{D}) = C_{\bar{T}}(\bar{K}\bar{D})$. Furthermore, $[K, R] = [K, V] \le V$, so each $\bar{K}_i$ is quasisimple. Thus $O_2(\bar{K}) \le Z(\bar{K})$. But $\bar{D}$ is $\bar{K}$-invariant and of odd order, so $O_2(\bar{K}\bar{D}) \le O_2(\bar{K})$, and as $O_2(\bar{K}\bar{D})$ centralizes $\bar{D}$, it follows that $O_2(\bar{K}\bar{D}) \le Z(\bar{K}\bar{D})$. Hence if we set $\bar{X}_1 = \bar{T}\bar{K}\bar{D}$, we see that Lemma 3.3 is applicable; we find that $V_1 = C_V(O_2(\bar{X}_1))$ is 2-reduced and singular in the preimage X_1 of $\bar{X}_1$ in G, and if $C_1 = C_{X_1}(V_1)$ and $\tilde{X}_1 = X_1/C_1$, then $F^*(\tilde{X}_1) = F(\tilde{D})$ and $\tilde{K}\tilde{D} \cong \bar{K}\bar{D}/O_2(\bar{K}\bar{D})$. But then $C_{\tilde{T}}(\tilde{D}) = C_{\tilde{T}}(\tilde{K}\tilde{D}) = 1$. Applying Proposition 3.4 to the preimage of $\tilde{T}\tilde{D}$, we see that $\langle \tilde{J}^{\tilde{D}} \rangle = \tilde{L}_1 \times \cdots \times \tilde{L}_r$, where $\tilde{L}_i \cong \Sigma_3$ and for each i, $\tilde{L}_i \cap \tilde{T} = \langle \tilde{a}_i \rangle$, where $\tilde{a}_i$ is a transvection on V_1. By Lemma 3.2(ii2), $\tilde{a}_i \in O_{2'2}(\tilde{X}_1)$. Hence $[\tilde{a}_i, \tilde{K}] = 1$ for each i, so $[\tilde{J}, \tilde{K}] = 1$. But $\tilde{K}$ is a central factor group of K^*, so $[J^*, K^*] = 1$. This contradicts $[J^*, K^*] = K^*$, and the lemma follows.

Now we complete the proof in a similar way. In the present case K^*D^* has odd order and so $C_{K^*D^*}(V) \le O_2(K^*)$. Thus $\bar{K}\bar{D} \cong K^*D^*$. Set $\bar{X}_1 = \bar{T}\bar{D}\bar{K}$, $T_1 = C_T(K^*D^*)$, $\bar{R} = O_2(\bar{X}_1)$, $V_1 = C_V(\bar{R})$, $C_1 = C_{X_1}(V_1)$ and $\tilde{X}_1 = X_1/C_1$. Set $\tilde{X}_2 = \langle \tilde{J}^{\tilde{X}_1} \rangle$. Again Proposition 3.4 gives $\tilde{X}_2 = \tilde{L}_1 \times \cdots \times \tilde{L}_r$, with each $\tilde{L}_i \cong \Sigma_3$. But $\tilde{K} = [\tilde{J}, \tilde{K}]$, by (8.3), so $\tilde{K}$ is the direct product of some of the $O_3(\tilde{L}_i)$. Now $O_3(\tilde{L}_i) = [O_3(\tilde{X}_2), \tilde{J} \cap \tilde{L}_i]$ is invariant under $C_{\tilde{K}\tilde{D}}(\tilde{J})$, and it follows that $\tilde{K}\tilde{D} = O_3(\tilde{X}_2) \times C_{\tilde{K}\tilde{D}}(\tilde{J})$. Therefore $[\tilde{K}, \tilde{D}] = 1$, and so $[K^*, D^*] = 1$. This contradicts Lemma 8.4 and completes the proof of Proposition 8.1.

As a corollary we obtain the following key restriction on the structure of C and $\bar{G}$.

PROPOSITION 8.6: *The following conditions hold:*

(i) $Q \in \mathrm{Syl}_2(C)$; *and*

(ii) *Either $F^*(\bar{G}) = F(\bar{G}) \le O(\bar{G})$ or $F^*(\bar{G}) = E(\bar{G})$ is a product of isomorphic components transitively permuted by $\bar{T}$, and $\bar{G} = E(\bar{G})\bar{T}$.*

Proof: Since $TC \leq C_G(Z) \leq G_0 < G$ and $C \triangleleft G$, Proposition 8.1 gives $Q \in \mathrm{Syl}_2(C)$, proving (i).

Since $O_2(\bar{G}) = 1$, either $F^*(\bar{G}) = F(\bar{G}) \leq O(\bar{G})$ or $E(\bar{G}) \neq 1$. Suppose $E(\bar{G})$ contains a nontrivial product $\bar{H}$ of components with $\bar{H} \triangleleft \bar{G}$ and $\bar{H}\bar{T} < \bar{G}$. If H denotes the preimage of $\bar{H}$ in G, then $H \triangleleft G$ and $HT < G$, whence $Q \in \mathrm{Syl}_2(H)$ by Proposition 8.1. But $Q \leq C$, so $\bar{Q} = 1$ and hence $\bar{H}$ is of odd order, contrary to the fact that $\bar{H}$ is a nontrivial product of components of $\bar{G}$. Hence no such $\bar{H}$ exists. It follows at once that either $F^*(\bar{G}) = F(\bar{G}) \leq O(\bar{G})$ or $F^*(\bar{G}) = E(\bar{G})$. In the latter case the components of $\bar{G}$ are isomorphic and transitively permuted by $\bar{G}$; and taking $E(\bar{G}) = \bar{H}$, we also deduce that $\bar{G} = E(\bar{G})\bar{T}$, so (ii) holds.

Finally, we sharpen the preceding proposition in case $F^*(\bar{G}) = F(\bar{G})$.

PROPOSITION 8.7: *If $F^*(\bar{G}) = F(\bar{G}) \leq O(\bar{G})$ and we set $\bar{L} = \langle \bar{J}^{\bar{G}} \rangle$, then $\bar{L} = \bar{L}_1 \times \bar{L}_2 \times \cdots \times \bar{L}_r$, where $\bar{L}_i \cong \Sigma_3$, $1 \leq i \leq r$, $\bar{T}$ permutes the $\bar{L}_i$ transitively, and $\bar{G} = \bar{L}\bar{T}$.*

Proof: Set $\bar{X} = O_{2'2}(\bar{G})\bar{J}$. Then the preimage X of $\bar{X}$ in G is solvable. Also $O_2(\bar{X}) = 1$ as $O(\bar{G}) \leq \bar{X}$ and $F^*(\bar{G}) \leq O(\bar{G})$. Hence if we set $\bar{L} = \langle \bar{J}^{\bar{X}} \rangle$, Proposition 3.4 implies that for some $r \geq 1$, $\bar{L} = \bar{L}_1 \times \cdots \times \bar{L}_r$, where $\bar{L}_i \cong \Sigma_3$ and $\langle \bar{a}_i \rangle = \bar{J} \cap \bar{L}_i \cong Z_2$ with $\bar{a}_i$ inducing a transvection on V, $1 \leq i \leq r$. In particular, $\bar{J} = \langle \bar{a}_i | 1 \leq i \leq r \rangle$. Hence by Lemma 3.2(ii), $\bar{J} \leq O_{2'2}(\bar{G})$.

Applying Proposition 8.1 to the preimage of $O_{2'2}(\bar{G})$, and noting that $\bar{J} \neq 1$, so that this preimage is not 2-closed, we conclude that $\bar{G} = O_{2'2}(\bar{G})\bar{T} = O_{2'2}(\bar{G}) = \bar{X}$. Thus $\bar{L} \triangleleft \bar{G}$. By the Krull-Schmidt Theorem, $\bar{T}$ permutes the $\bar{L}_i$. If $\bar{L}_0$ is a nontrivial $\bar{T}$-invariant product of $\bar{L}_i$, then again Proposition 8.1 gives a contradiction if $\bar{L}_0\bar{T} < \bar{G}$. Therefore $\bar{G} = \bar{L}_0\bar{T}$, whence $\bar{T}$ permutes the $\bar{L}_i$ transitively, and the result is proved.

9. The reduction step

We preserve the assumptions and notation of the preceding section. Then G has a singular 2-reduced setup with $V = \langle Z^G \rangle$ and Propositions 8.6 and 8.7 give the structure of $\bar{G}$ as well as the fact that $Q \in \mathrm{Syl}_2(C)$.

In this section we eliminate all but three possibilities for the structure of $\bar{G}$, proving

PROPOSITION 9.1: *One of the following holds:*

(i) $\bar{G} \cong L_2(2^n)$, $n \geq 1$;

(ii) $\bar{G} \cong \Sigma_{2n+1}$, $n \geq 2$, *and* G *contains a near component of type* A_{2n+1}; *or*
(iii) $\bar{G} \cong A_7$ *and* G *contains a near component of small* A_7 *type.*

We argue in a sequence of lemmas. Since G has a singular 2-reduced setup, $\bar{J} \neq 1$. Set $\bar{L} = E(\bar{G})$, so that either $\bar{L} = 1$ or $\bar{L}$ is the product of isomorphic components $\bar{L}_i$, $1 \leq i \leq r$, transitively permuted by $\bar{T}$, and $\bar{G} = \bar{L}\bar{T}$ by Proposition 8.6. Since $\bar{J} \triangleleft \bar{T}$, it follows in the latter case that $\bar{J}$ centralizes no $\bar{L}_i$, $1 \leq i \leq r$. On the other hand, if $F^*(\bar{G}) = F(\bar{G}) \leq O(\bar{G})$, then by Proposition 8.7, we know that $\bar{G} = \bar{L}\bar{T}$, where $\bar{L}$ is the direct product of subgroups $\bar{L}_i \cong \Sigma_3(\cong L_2(2))$, $1 \leq i \leq r$, transitively permuted by $\bar{T}$ with $\bar{G} = \bar{L}\bar{T}$ and $\bar{L} = \langle \bar{J}^{\bar{L}} \rangle$.

In either case, $\bar{G}_0$ contains no $\bar{L}_i$, for if it did, then since $\bar{G}_0 \geq \bar{T}$, $\bar{G}_0 \geq \langle \bar{T}, \bar{L}_i \rangle = \bar{G}$, and since $C \leq G_0$, $G_0 = G$, contradiction.

Furthermore, we choose $A \in \mathcal{A}(T)$ such that $\bar{A} \neq 1$ and subject to this, so that $\bar{A}$ has minimal order. Then if $B \in \mathcal{A}(T)$ and $1 \neq \bar{B} \leq \bar{A}$, we have $\bar{B} = \bar{A}$. Since $[\bar{A}, \bar{L}_j] \neq 1$ for some j, we can find, in view of the transitive action of $\bar{T}$, $\bar{T}$-conjugates $\bar{A}_i$ of $\bar{A}(1 \leq i \leq r)$, such that $[\bar{A}_i, \bar{L}_i] \neq 1$ for each i, and the $\bar{A}_i$ are also minimal. This choice, together with Propositions 8.1, 4.1, and 3.4(iv), enables us to apply Propositions 6.1 and 7.1 to any $\bar{L}_i \cong L_2(2^n)$ or A_{2n+1}, as we shall do in the proofs of Lemmas 9.5 and 9.6.

First of all, since $[\bar{J}, \bar{L}_1] \neq 1$, Proposition 3.5 immediately eliminates many possibilities for components of $\bar{G}$, yielding

LEMMA 9.2: *If* $E(\bar{G}) \neq 1$, *then the following conditions hold:*
(i) $\bar{L}_1 \in Chev(2) \cup Alt \cup Spor$; *and*
(ii) $\bar{L}_1 \not\cong Sz(2^n)$, $(S)U_3(2^n)$, $3A_7$, Ly, *or* J_1.

Similarly we prove

LEMMA 9.3: *If* $\bar{L}_1/Z(\bar{L}_1) \cong L_3(2^n)$ *or* $Sp_4(2^n)'$, *then the image of* T *in* $\mathrm{Out}(\bar{L}_1)$ *acts trivially on the Dynkin diagram of* $\bar{L}_1$.

Proof: Suppose false. By Lemma 4.2, $\bar{J}$ leaves $\bar{L}_1$ invariant. We set $\bar{X} = \bar{L}_1\bar{J}\langle \bar{t} \rangle$, where $\bar{t} \in \bar{T}$ induces a graph or graph-field automorphism on $\bar{L}_1$, make the usual reduction with Lemma 3.3, and then apply Proposition 3.13. Since $\bar{J}$ does not centralize $\bar{L}_1$, it yields a contradiction.

We can now prove

LEMMA 9.4: *Either* $\bar{L}_1 \cong L_2(2^n)$, $n \geq 1$, *or* $\bar{L}_1 \cong A_{2n+1}$, $n \geq 2$.

Proof: If $E(\bar{G}) = 1$, then $\bar{L}_1 \cong L_2(2) \cong \Sigma_3$, as noted at the beginning of the section. Hence we can suppose that $E(\bar{G}) \neq 1$. We assume the lemma fails. If $\bar{L}_1 \in Chev(2)$, then $\bar{L}_1 \not\cong L_2(2^n)$, $Sz(2^n)$, or $(S)U_3(2^n)$ (Lemma 9.2(ii))) and if $\bar{L}_1/Z(\bar{L}_1) \cong L_3(2^n)$ or $Sp_4(2^n)'$, then no element of $\bar{T}$ induces a graph or graph-field automorphism on $\bar{L}_1$ (Lemma 9.3). If $\bar{L}_1 \in Alt - Chev(2)$, then $\bar{L}_1 \cong A_{2n}$, $n \geq 5$, and if $\bar{L}_1 \in Spor$, then $\bar{L}_1 \not\cong J_1$ or Ly (Lemma 9.2(ii)). Hence if we set $\bar{T}_1 = N_{\bar{T}}(\bar{L}_1)$, Proposition 5.1 implies that $\bar{L}_1$ is generated by $\bar{T}_1$-invariant subgroups $\bar{H}_i$, $1 \leq i \leq m$, such that $O(\bar{H}_i) \leq Z(\bar{L}_1)$ and no component of $\bar{H}_i$ is isomorphic to $L_2(2^n)$ or A_{2^n+1} for any $n \geq 2$.

Let $\bar{t}_j$, $1 \leq j \leq r$, be representatives of the distinct cosets of $\bar{T}_1$ in $\bar{T}$ with $\bar{t}_1 = 1$ and $\bar{L}_1^{\bar{t}_j} = \bar{L}_j$, $1 \leq j \leq r$. Then the corresponding assertion holds for $\bar{L}_j$ relative to the subgroups $\bar{H}_i^{\bar{t}_j}$ and $\bar{T}_1^{\bar{t}_j}$. Hence if we set $\bar{I}_i = \bar{H}_i^{\bar{t}_1}\bar{H}_i^{\bar{t}_2}\cdots\bar{H}_i^{\bar{t}_r}$, we conclude that

(9.1)
(1) $\bar{L} = \langle \bar{I}_i \,|\, 1 \leq i \leq m\rangle$, and for each i:
(2) $\bar{I}_i$ is $\bar{T}$-invariant;
(3) $O(\bar{I}_i) \leq Z(\bar{L})$; and
(4) No component of $\bar{I}_i$ is isomorphic to $L_2(2^n)$ or A_{2n+1} for $n \geq 2$.

Let I_i be the preimage of $\bar{I}_i$ in G, $1 \leq i \leq m$, so that each I_i is T-invariant with $I_iT < G$. It follows that each I_i has Aschbacher form. If each $\bar{I}_i \leq \bar{G}_0$, then $\bar{L} = \langle \bar{I}_i \,|\, 1 \leq i \leq m\rangle \leq \bar{G}_0$, contrary to Proposition 8.6(iii). For definiteness, suppose $\bar{I}_1 \not\leq \bar{G}_0$, whence I_1 possesses a near component K of type $L_2(2^n)$, $n \geq 1$, or A_{2m+1}, $m \geq 2$, with $\bar{K} \not\leq \bar{G}_0$ as I_1 has Aschbacher form.

Since $\bar{K} \not\leq \bar{G}_0$ and $Z \leq V$, $[K, V] \neq 1$, whence $[K, O_2(K)] \leq V$. Moreover, either $\bar{K} \cong Z_3$, $L_2(2^n)$, $n \geq 2$, or A_{2m+1}. However, in the first case it follows from (9.1) that $\bar{K} \leq Z(\bar{L}) \leq \bar{L}'$. But $[K, Q] \cong E_4$ in this case as $Q \leq I_1$ and K is a near component of I_1. This is impossible as $\bar{L}$ leaves $[K, Q] = [K, V]$ invariant and $\bar{K} \leq \bar{L}'$. We conclude that $\bar{K}$ is quasisimple of one of the specified types. But $\bar{K}$ is subnormal in $\bar{I}_1$, whence $\bar{K}$ is a component of $\bar{I}_1$. This violates (9.1)(4), so the lemma is proved.

We treat the two possibilities of the lemma separately, first proving

LEMMA 9.5: *If* $\bar{L}_1 \cong L_2(2^n)$, $n \geq 1$, *then* $\bar{G} = \bar{L}_1 \cong L_2(2^n)$.

Proof: By choice, $\bar{L}_1$ is subnormal in $\bar{G}$, so by Proposition 6.1, for each i, $1 \le i \le r$, G contains an S-invariant subgroup Y_i such that

(9.2)
(1) $Y_i \cap S \in \mathrm{Syl}_2(Y_i)$;
(2) $\bar{Y}_i = \bar{L}_i'$ and $\bar{S}$ induces inner automorphisms on $\bar{L}_i$;
(3) $F^*(Y_i) = O_2(Y_i)$; and
(4) Y_i is subnormal in G.

Note that if $r = 1$ and $\bar{L}_1\bar{S} = \bar{G}$, then as $\bar{S}$ induces inner automorphisms on $\bar{L}_1$, it follows that $\bar{L}_1 = \bar{G}$, as asserted. Hence we can assume that either $r > 1$ or $\bar{L}_1\bar{S} < \bar{G}$. In either case, this implies that $Y_iS < G$, $1 \le i \le r$. As observed at the beginning of §8, we conclude now that each Y_i has Aschbacher form (relative to S). Since $C(Y_i; S) \le G_0$ (as S char T), it follows that $Y_i = K_i(Y_i \cap G_0)$, where $K_i \triangleleft Y_i$ and either $K_i = 1$ or K_i is a near component of type $L_2(2^n)$. Since Y_i is subnormal in G, so is K_i and hence either $K_i = 1$ or K_i is a near component of G.

Now $\bar{L}_i' = O^2(\bar{Y}_i) \not\le \bar{G}_0$, so $\bar{L}_i' = \bar{K}_i'$ for each i. But $\bar{G} = \bar{L}\bar{T} = \bar{L}'\bar{T}$, so $\bar{G} = \bar{K}_1 \cdots \bar{K}_r\bar{G}_0$ as $T \le G_0$. Since $C \le G_0$, we have $G = K_1 \cdots K_rG_0$, whence G has Aschbacher form, contradiction.

Finally we prove

LEMMA 9.6: *If $\bar{L}_1 \cong A_{2n+1}$, $n \ge 2$, then $\bar{G} \cong \Sigma_{2n+1}$ or A_7 and correspondingly G contains a near component of type A_{2n+1} or small A_7 type.*

Proof: Let $\bar{I}_1$ be a subgroup of $\bar{L}_1$ isomorphic to A_{2n} and invariant under $\bar{T}_1 = N_{\bar{T}}(\bar{L}_1)$, and let $\bar{I}_i$ be a $\bar{T}$-conjugate of $\bar{I}_1$ contained in $\bar{L}_i$, $1 \le i \le r$. Then $\bar{I} = \bar{I}_1\bar{I}_2 \cdots \bar{I}_r$ is $\bar{T}$-invariant. Also clearly $\bar{I}\bar{T}$ contains no components isomorphic to $L_2(2^m)$ or A_{2m+1}, and $O(\bar{I}) = 1$. Furthermore, if we let I be the preimage of $\bar{I}$ in G, then I is T-invariant with $Q \le I$ and $IT < G$, so I has Aschbacher form. However, as in the proof of Lemma 9.4, the structure of $\bar{I}$ implies that I contains no near components of the required types, whence $I \le G_0$. Thus $\bar{I} \le \bar{G}_0$.

On the other hand, Proposition 7.1 yields that for each i, $1 \le i \le r$, G contains an S-invariant subgroup Y_i such that

(9.3)
(1) $Y_i \cap S \in \mathrm{Syl}_2(Y_i)$;
(2) $\bar{Y}_i \cong \Sigma_3$ or A_5;
(3) $F^*(Y_i) = O_2(Y_i)$; and
(4) $\bar{L}_i \le \langle \bar{I}_i, \bar{Y}_i \rangle$.

However, since $\bar{L}_i \not\leq \bar{G}_0$ and $\bar{I}_i \leq \bar{G}_0$, it follows that $\bar{Y}_i \not\leq \bar{G}_0$ for each i. But $C \leq G_0$, so $Y_i \not\leq G_0$, $1 \leq i \leq r$. Clearly $Y_iS < G$, so Y_iS has Aschbacher form and consequently $Y_i = H_iC(Y_i; S)$, where H_i is a near component of Y_i of type $L_2(2)$ or $L_2(4)$.

But now if we let L_i be the preimage of $\bar{L}_i$ in G, we conclude from Proposition 7.1(ii) that for each i, $1 \leq i \leq r$,

(9.4)
(1) $L_i = K_iC$, where K_i is a near component of $L_i$of type A_{2n+1} or of small A_7 type; and
(2) T contains a characteristic subgroup P_i such that P_i leaves K_i invariant and $P_i \cap K_i \in \mathrm{Syl}_2(K_i)$.

If $r = 1$, then as G has a singular 2-reduced setup, $\bar{G} = \bar{L}_1\bar{T} \cong \Sigma_{2n+1}$ or $\bar{G} = A_7$, and the lemma holds. We can therefore assume that $r > 1$, whence each $K_iP_i < G$. But then K_iP_i has Aschbacher form (relative to P_i), so either $K_i = C(K_i; P_i) \leq G_0$ (as P_i char T) or else $2n + 1 = 2^m + 1$ for some m. The first case is impossible since $\bar{K}_i = \bar{L}_i \not\leq \bar{G}_0$, so the second holds. Therefore $G = K_1K_2 \cdots K_rG_0$ has Aschbacher form, again a contradiction, and the lemma is proved.

This completes the proof of Proposition 9.1.

10. The residual configurations

We eliminate the three possibilities of Proposition 9.1 by showing that G has Aschbacher form, contrary to the choice of G. We preserve the preceding notation.

We first prove

PROPOSITION 10.1: *If $\bar{G} \cong L_2(2^n)$, $n \geq 1$, then G has Aschbacher form.*

Proof: Set $G^* = G/Q$. Since $Q \in \mathrm{Syl}_2(C)$ by Proposition 8.1, $|C^*|$ is odd and $G^*/C^* \cong \bar{G}$. Furthermore, $G_0 < G$, so no nontrivial characteristic subgroup of T is normal in G. Hence if $C^* = \Phi(G^*)$, Theorem 1.1 implies that $O^2(G)$ is a near component of type $L_2(2^n)$, so G has Aschbacher form, as desired. We may thus assume that $C^* \neq \Phi(G^*)$.

As G^*/C^* is either simple or isomorphic to $L_2(2)(\cong \Sigma_3)$, $\Phi(G^*/C^*) = 1$, so $\Phi(G^*) < C^*$. However, as $C^* \triangleleft G^*$, it is clear that every maximal subgroup of G^* either covers G^*/C^* or else contains C^*, so some maximal subgroup M^* of

G^* must cover G^*/C^*. Moreover, as $|C^*|$ is odd, we can choose M^* to contain T^*. Let M be the preimage of M^* in G, so that $T \leq M < G$ and $G = MC$. In particular, since $C \leq G_0$, $M \not\leq G_0$. It follows therefore from the minimality of G that $M = KM_0$, where K is normal near component of M of type $L_2(2^n)$ and $K \not\leq G_0$.

Finally, $G = MC = KG_0$ and $\bar{K} \cong L_2(2^n)'$. Also $[K, Q]$ has one nontrivial K-chief factor and $[K, V] \neq 1$, so $[K, Q] \leq V$. Set $\tilde{G} = G/V$, so that $\tilde{K}$ centralizes $\tilde{Q}$. Put $\tilde{H} = \tilde{K}\tilde{C}$. Since $\tilde{C}$ centralizes V and $C_G(Q) \leq Q$, $C_{\tilde{C}}(\tilde{Q}) \leq \tilde{Q}$, so $C_{\tilde{H}}(\tilde{Q}) = \tilde{K}Z(\tilde{Q})$, whence $O^2(C_{\tilde{H}}(\tilde{Q})) = \tilde{K}$. But $C_{\tilde{H}}(\tilde{Q}) \triangleleft \tilde{H}$, so $\tilde{K} \triangleleft \tilde{H}$. Thus $[\tilde{K}, \tilde{C}] \leq \tilde{K} \cap \tilde{C} \triangleleft \tilde{K}$. However, $\tilde{C}$ is solvable, so $\tilde{K} \cap \tilde{C} \leq Z(\tilde{K})$. Hence if $\tilde{K}$ is nonsolvable, the three subgroups lemma yields that $\tilde{K}$ centralizes $\tilde{C}$. On the other hand, if $\tilde{K} \cong Z_3$, then as $K \not\leq G_0$, $\tilde{K} \cap \tilde{C} = 1$, so $\tilde{K}$ centralizes $\tilde{C}$ in this case as well.

Thus $KV \triangleleft KC$. But $O^2(KV) = K$, so $K \triangleleft KC$. Since $G = MC$ and $K \triangleleft M$, $K \triangleleft G$, so K is a normal near component of G. Hence as $G = KG_0$, again G has Aschbacher form, and the proposition is proved.

PROPOSITION 10.2: *If $\bar{G} \cong \Sigma_{2n+1}$, $n \geq 2$, or A_7, and K is a near component of G of A_{2n+1} type or small A_7 type, respectively, then G has Aschbacher form.*

Proof: We have $G = KCT = KG_0$. If $2n+1 = 2^m + 1$, then G has Aschbacher form, so we may assume that $2n + 1 \neq 2^m + 1$. Set $X = KT$. Thus $G = XG_0$. If K is of small A_7 type, we apply Lemma 5.3 to X and conclude that $X \leq G_0$, so $G = G_0$, as needed. Otherwise we apply Proposition 5.2 to X and conclude that $X \leq \langle H, G_0 \rangle$, where $H \leq X$, H is T-invariant, and $\bar{H} \cong A_{2n}$. As argued in the proof of Proposition 9.1, $H \leq G_0$, so again $X \leq G_0$ and $G = G_0$, and the proposition follows.

Propositions 9.1, 10.1, and 10.2 now yield Theorem A.

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